Playing around with the gamma distribution

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1 Generalities

- Here is discussed the gamma distribution used as the probability density function of number concentration of a hydrometeor per its diameter. It is a function of three parameters: the number concentration of a hydrometeor N, shape parameter μ, and slope parameter λ. A single-moment microphysics scheme has prognostical only λ. A double-moment scheme also has prognostical N, but μ still must be prescribed (or diagnosed).
- ICE3 and LIMA use the *generalized* gamma distribution with one more parameter, which is not discussed in this document, but one can redo the computations with it.
- The gamma distribution is defined as:

$$n(D) = N\rho_t \frac{\lambda^{\mu+1}}{\Gamma(\mu+1)} D^{\mu} e^{-\lambda D}\,[m^{-4}], \tag{1}$$

where $N\,[\mathrm{kg}^{-1}]$ is the total number of particles of a given hydrometeor, $\rho_t\,[\mathrm{kg{\cdot}m^{-3}}]$ is the density of air with water species, $D\,[\mathrm{m}]$ diameter of the hydrometeor, μ is a dimensionless shape parameter, and $\lambda\,[\mathrm{m}^{-1}]$ is the slope parameter. There is ρ_t since N is in kg^{-1}. The shape parameter $\mu \in \mathbb{R}, \mu \geq 0$ to keep things simple as $\mu < 0$ is not widely used.

- Suppose $r \in \mathbb{R}$ such that $r \geq 0$. Then the r-th moment of the gamma distribution is:

$$\begin{aligned} M_r = \int_0^{\infty} D^r n(D) dD &= N\rho_t \frac{\lambda^{\mu+1}}{\Gamma(\mu+1)} \int_0^{\infty} D^{\mu+r} e^{-\lambda D} dD \\ &= N\rho_t \frac{\lambda^{\mu+1}}{\Gamma(\mu+1)} \frac{\Gamma(\mu+1+r)}{\lambda^{\mu+1+r}} \\ &= N\rho_t \frac{\Gamma(\mu+1+r)}{\Gamma(\mu+1)} \lambda^{-r}. \end{aligned} \tag{2}$$

- The value of λ is obtained from the mass fraction $q\,[\mathrm{kg{\cdot}kg^{-1}}]$:

$$q = \frac{1}{\rho_t}\int_0^{\infty} m(D) n(D) dD = \frac{1}{\rho_t}\int_0^{\infty} \frac{\rho\pi D^3}{6} N\rho_t \frac{\lambda^{\mu+1}}{\Gamma(\mu+1)} D^{\mu} e^{-\lambda D} = \frac{\rho\pi N \Gamma(\mu+4)}{6\Gamma(\mu+1)} \frac{1}{\lambda^3} \tag{3}$$

$$\lambda = \left[\frac{\rho\pi N\Gamma(\mu+4)}{6q\Gamma(\mu+1)}\right]^{\frac{1}{3}}, \tag{4}$$

where $m\,[\mathrm{kg}]$ is the mass of the hydrometeor and $\rho\,[\mathrm{kg{\cdot}m^{-3}}]$ its density.

2 Diameters and their ratios

2.1 Definitions

- The *mean volume diameter* is the diameter for a monodisperse size distribution. In other words, if all drops are of the same size at given q and N, then the mean volume diameter is their diameter. Mathematically speaking:

$$D_V = \left(\frac{6q}{\pi\rho N}\right)^{\frac{1}{3}}. \tag{5}$$

- The *effective diameter* is weighted by D^2, with proportional to the surface of the hydrometeor:

$$D_{eff} = \frac{\int_0^{\infty} D^3 n(D) dD}{\int_0^{\infty} D^2 n(D) dD} = \frac{\Gamma(\mu+4)}{\Gamma(\mu+3)}\frac{1}{\lambda} = \frac{\Gamma(\mu+4)}{\Gamma(\mu+3)}\left[\frac{6q\Gamma(\mu+1)}{\rho\pi N\Gamma(\mu+4)}\right]^{\frac{1}{3}} = \frac{\Gamma(\mu+4)}{\Gamma(\mu+3)}\left[\frac{\Gamma(\mu+1)}{\Gamma(\mu+4)}\right]^{\frac{1}{3}} D_V. \tag{6}$$

- The *mass-weighted mean diameter* is weighted with the mass, which is proportional to the third power of diameter for a spherical particle, so it is defined for a spherical particle as:

$$D_m = \frac{\int_0^\infty D m(D) n(D) dD}{\int_0^\infty m(D) n(D) dD} = \frac{\int_0^\infty D^{\mu+4} e^{-\lambda D} dD}{\int_0^\infty D^{\mu+3} e^{-\lambda D} dD} = \frac{\Gamma(\mu+5)}{\Gamma(\mu+4)} \left[\frac{6q\Gamma(\mu+1)}{\rho\pi N\Gamma(\mu+4)}\right]^{\frac{1}{3}} = \frac{\Gamma(\mu+5)}{\Gamma(\mu+4)} \left[\frac{\Gamma(\mu+1)}{\Gamma(\mu+4)}\right]^{\frac{1}{3}} D_V. \quad (7)$$

- There are other diameters (e.g. number-concentration-weighted mean diameter or median diameter), but they are probably not so useful at the moment.

2.2 Ratios

- The ratio of D_{eff} to D_V is:

$$\frac{D_{eff}}{D_V} = \frac{\Gamma(\mu+4)}{\Gamma(\mu+3)} \left[\frac{\Gamma(\mu+1)}{\Gamma(\mu+4)}\right]^{\frac{1}{3}} = \frac{\mu+3}{[(\mu+3)(2+\mu)(\mu+1)]^{\frac{1}{3}}} > 1. \quad (8)$$

  The relationship $\Gamma(x+1) = x\Gamma(x),\ x > 0$ was used for simplification[1]. The ratio is show in Figure 1.

- The ratio of D_m to D_V is (also see Figure 1):

$$\frac{D_m}{D_V} = \frac{\Gamma(\mu+5)}{\Gamma(\mu+4)} \left[\frac{\Gamma(\mu+1)}{\Gamma(\mu+4)}\right]^{\frac{1}{3}} = \frac{\mu+4}{[(\mu+3)(2+\mu)(\mu+1)]^{\frac{1}{3}}} > 1. \quad (9)$$

- Finally, the ratio of D_m to D_{eff} is (again see Figure 1):

$$\frac{D_m}{D_{eff}} = \frac{D_m}{D_V}\frac{D_V}{D_{eff}} = \frac{\Gamma(\mu+5)}{\Gamma(\mu+4)}\frac{\Gamma(\mu+3)}{\Gamma(\mu+4)} = \frac{(\mu+4)(\mu+3)\left[\Gamma(\mu+3)\right]^2}{\left[(\mu+3)\Gamma(\mu+3)\right]^2} = \frac{\mu+4}{\mu+3} > 1. \quad (10)$$

- $D_V < D_{eff} < D_m,\ \forall \mu \in \mathbb{R} : \mu \geq 0.$

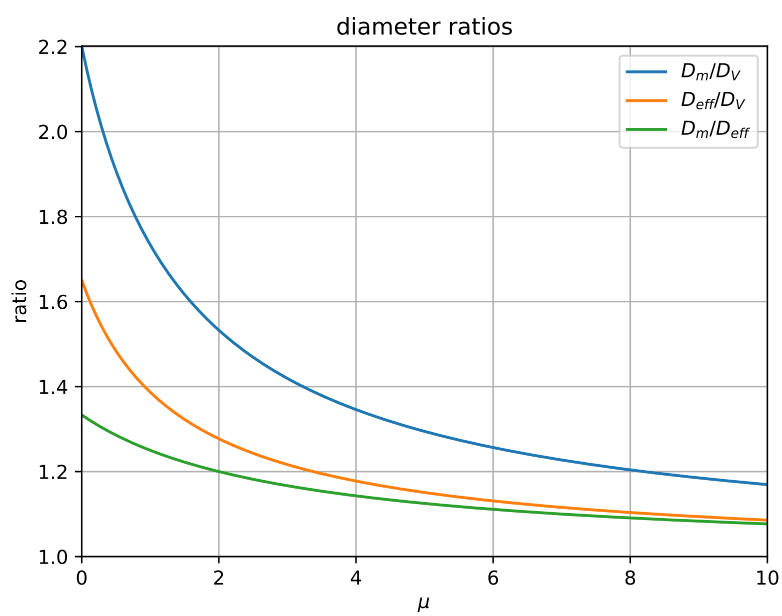

Figure 1: Ratio of the D_m to D_V dependency on the shape parameter.

[1] Thank you to Ján Mašek for suggesting using this relationship.