New semi-Lagrangian interpolators implemented in IFS/ARPEGE/ALADIN

Ján Mašek, Slovak HydroMeteorological Institute

Historical background (1)

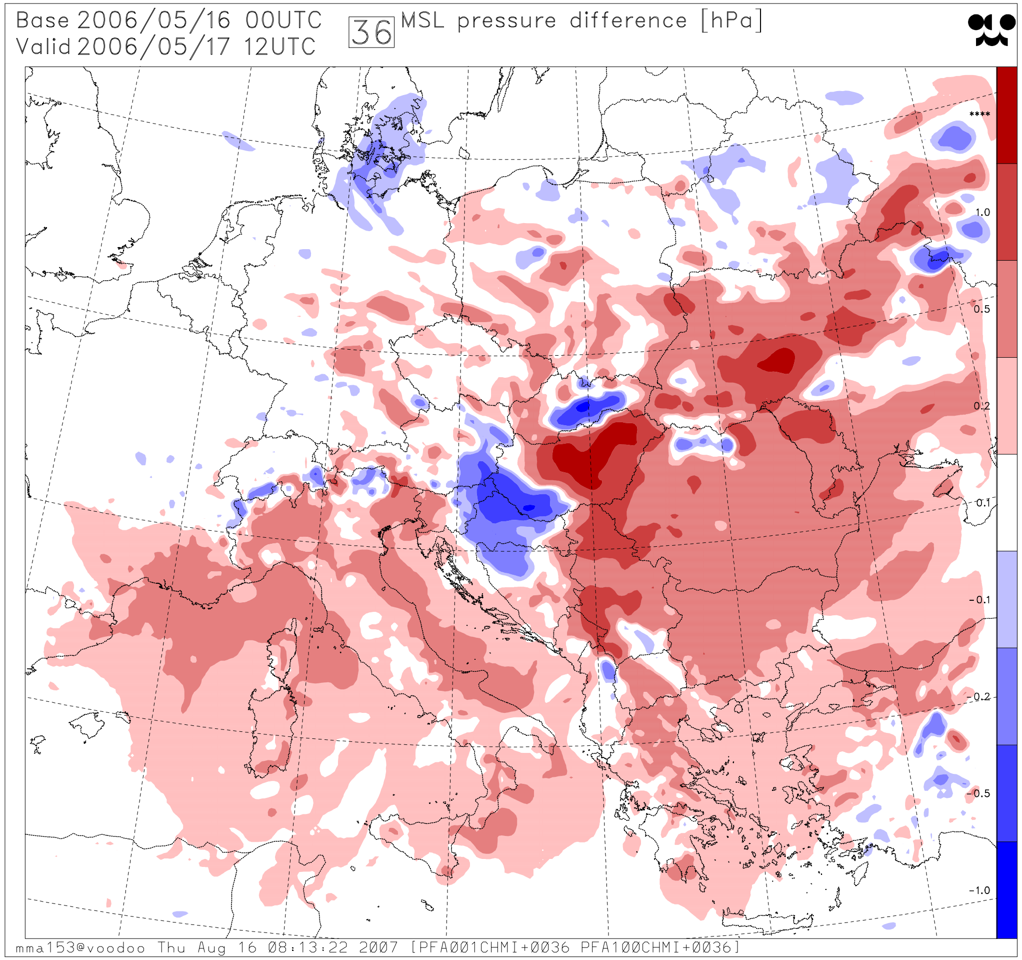
- SLHD scheme (Semi-Lagrangian Horizontal Diffusion) implements nonlinear diffusion by exploiting damping properties of semi-Lagrangian interpolations

- diffusivity of semi-Lagrangian interpolations is controlled by mixing accurate high order interpolator with diffusive low order one

- disadvantage of such approach is reduced accuracy, resulting in worsened mass conservation seen as positive MSLP bias

Mass (non)conservation

old SLHD scheme, MSLP difference against reference without SLHD
+36 h forecast



Historical background (2)

- there was an attempt to improve SLHD conservative properties by changing high order semi-Lagrangian interpolator

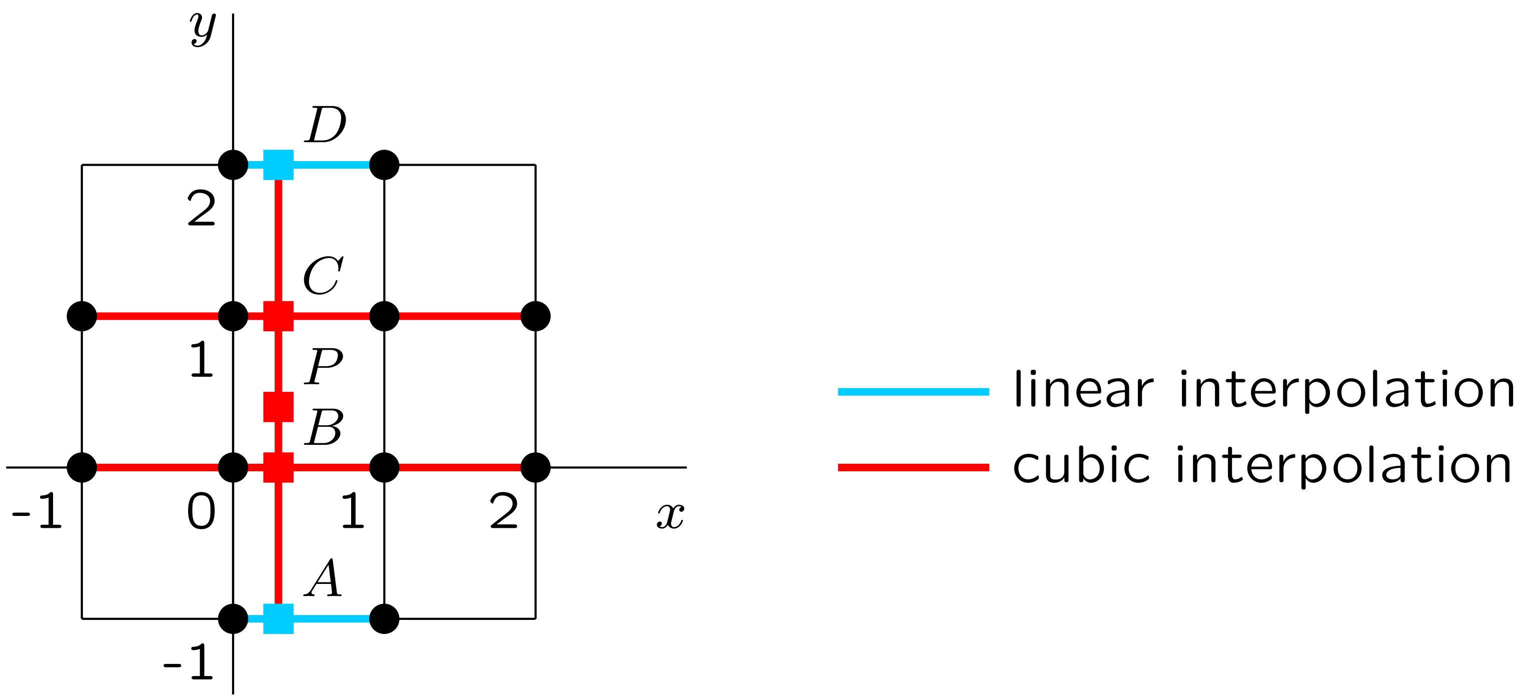
- cubic Lagrange polynomial was replaced with more accurate natural cubic spline

- it indeed reduced positive MSLP bias, but detrimental effect on other fields was seen

- in order to understand what is going on, detailed examination of semi-Lagrangian interpolators followed

From 2D/3D interpolators down to 1D interpolators

- high order 2D interpolators in model are composed of high and low order 1D interpolators, acting on 12-point stencil:



- similar approach is used in 3D case, employing 32-point stencil

Requirements on decent 4-point 1D interpolator

1. linearity with respect to $\mathbf{y} = (y_{-1}, y_0, y_1, y_2)$:

$$F(x, \mathbf{y}) = w_{-1}(x)y_{-1} + w_0(x)y_0 + w_1(x)y_1 + w_2(x)y_2$$

2. invariace with respect to horizontal mirroring:

$$F(1-x, y_2, y_1, y_0, y_{-1}) = F(x, y_{-1}, y_0, y_1, y_2)$$

3. invariace with respect to vertical shift:

$$F(x, \mathbf{y} + c) = F(x, \mathbf{y}) + c$$

4. reproducing of values y_0, y_1:

$$F(0, \mathbf{y}) = y_0$$
$$F(1, \mathbf{y}) = y_1$$

5. reproducing of linear function $y = x$:

$$F(x, -1, 0, 1, 2) = x$$

Family of cubic 4-point 1D interpolators

- when weights w_{-1}, w_0, w_1, w_2 are constrained to polynomials of degree at most 3, interpolator F is restricted to the form:

$$
\begin{aligned}
F(x, \mathbf{y}) &= u(x)y_{-1} + v(x)y_0 + v(1-x)y_1 + u(1-x)y_2 \\
u(x) &= a_1 x + a_2 x^2 - (a_1 + a_2)x^3 \\
v(x) &= 1 + (a_2 - 1)x - (3a_1 + 4a_2)x^2 + 3(a_1 + a_2)x^3
\end{aligned}
$$

$$a_1, a_2 \in \mathbb{R}$$

- every decent 4-point cubic interpolator can be represented by point in (a_1, a_2) plane

- requirement that F reproduces also quadratic function $y = x^2$ (which implies second order accuracy) defines straight line:

$$6a_1 + 2a_2 = -1$$

Family relations and important family members

- every second order accurate 4-point 1D interpolator F can be written as weighted combination of cubic Lagrange polynomial F_{lag} and quadratic interpolator F_{quad}:

$$F = (1-\kappa)F_{\text{lag}} + \kappa F_{\text{quad}} \qquad \kappa \in \mathbb{R}$$

- identity of some important inhabitants of (a_1, a_2) plane:

address		name	order of accuracy	parameter κ
a_1	a_2			
0	0	linear interpolator	1	——
$-\frac{1}{4}$	$\frac{1}{4}$	quadratic interpolator	2	1
$-\frac{1}{3}$	$\frac{1}{2}$	cubic Lagrange polynomial	3	0
$-\frac{1}{2}$	1	quasi-cubic spline	2	-2
$-\frac{7}{15}$	$\frac{4}{5}$	natural cubic spline	1	——

How to measure accuracy?

- accuracy of (a_1, a_2) interpolators was evaluated on sample of harmonic test functions:

$$y_m(x) = \sin(2\pi m x/N) \qquad x \in [0, N] \qquad m = 1, 2, \ldots, M$$

- source grid had $N = 100$ intervals, linear truncation with $M = 49$ was chosen

- each test function was interpolated onto 20 times finer target grid

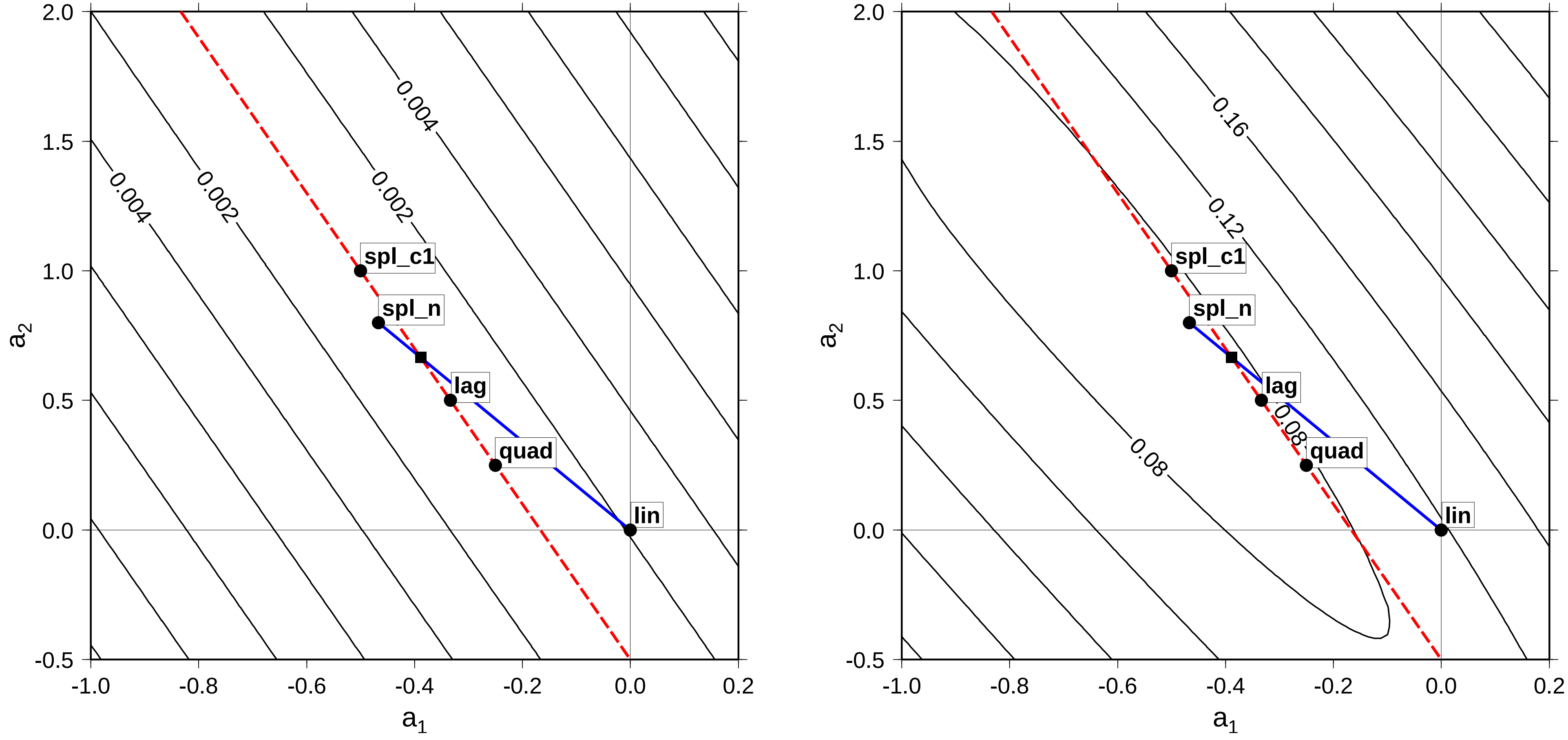
- overall accuracy of interpolator was measured by MAE weighted by function $\exp(-\beta m/M)$

- parameter β was used to control significance of short waves

Accuracy maps

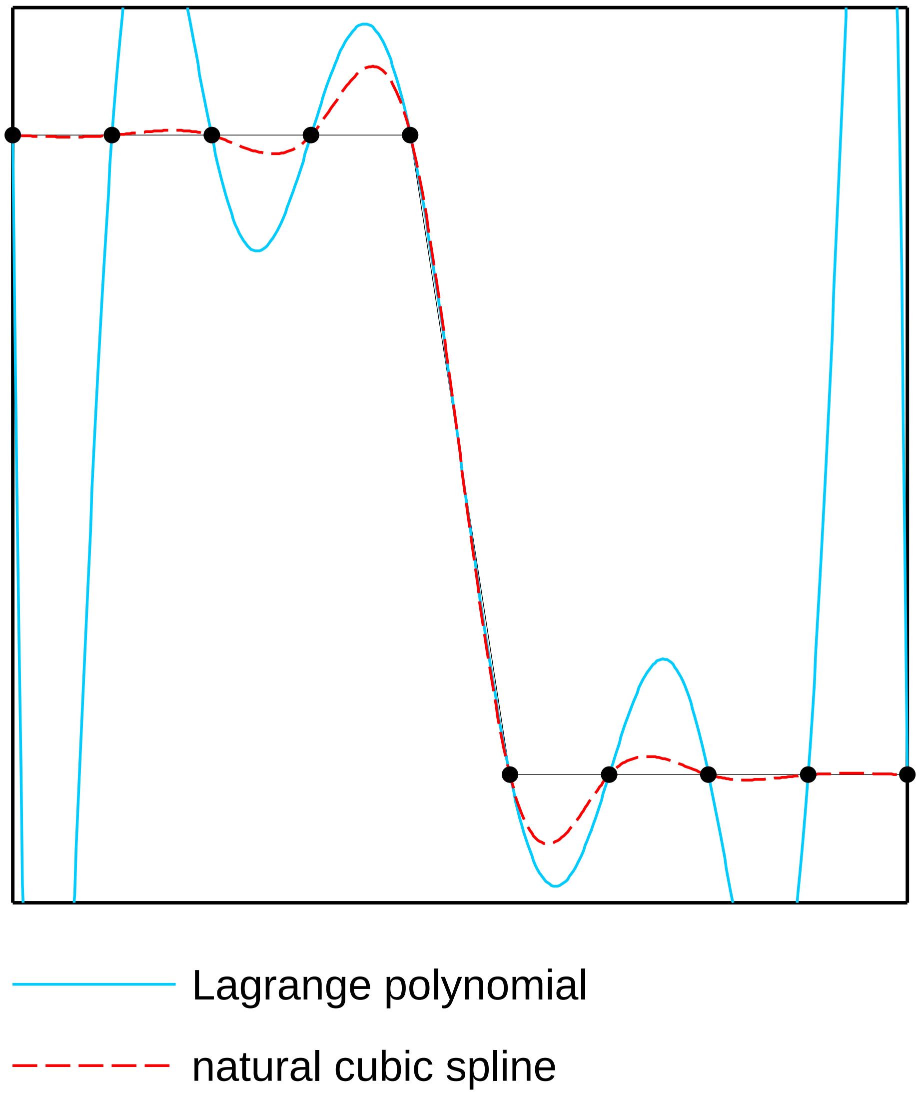
accuracy measured by weighted MAE

weight function $\exp(-25m/M)$ | weight function $\exp(-m/M)$

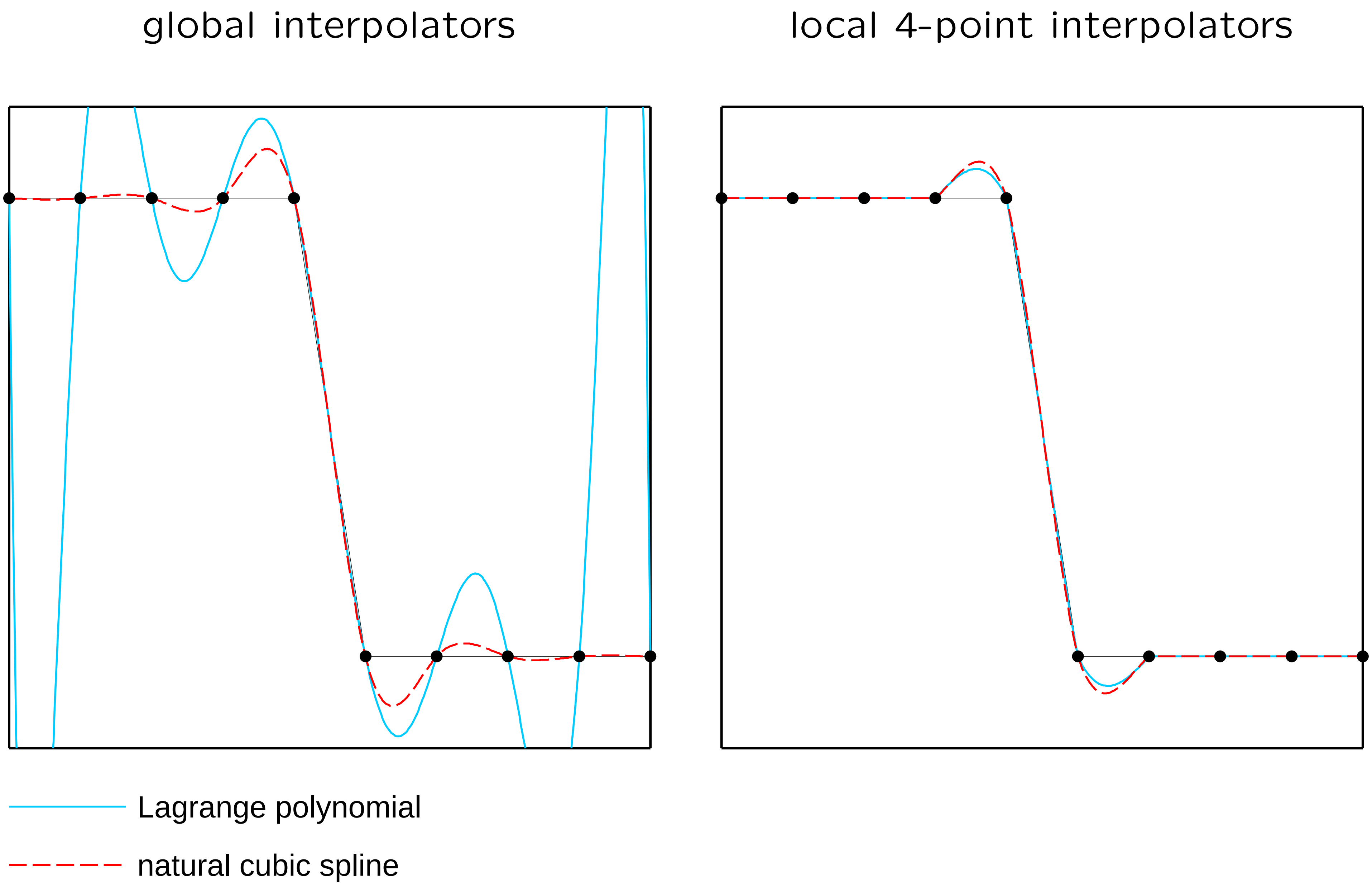


Overshoots – global versus local interpolators

global interpolators



Overshoots – global versus local interpolators



How to measure diffusivity?

- linear advection of harmonic wave performed by semi-Lagrangian scheme:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

$$u(x_n, t_0) = \sin(kx_n)$$

$$u(x_n, t_i) = u(x_n - c\Delta t_i, t_{i-1})$$

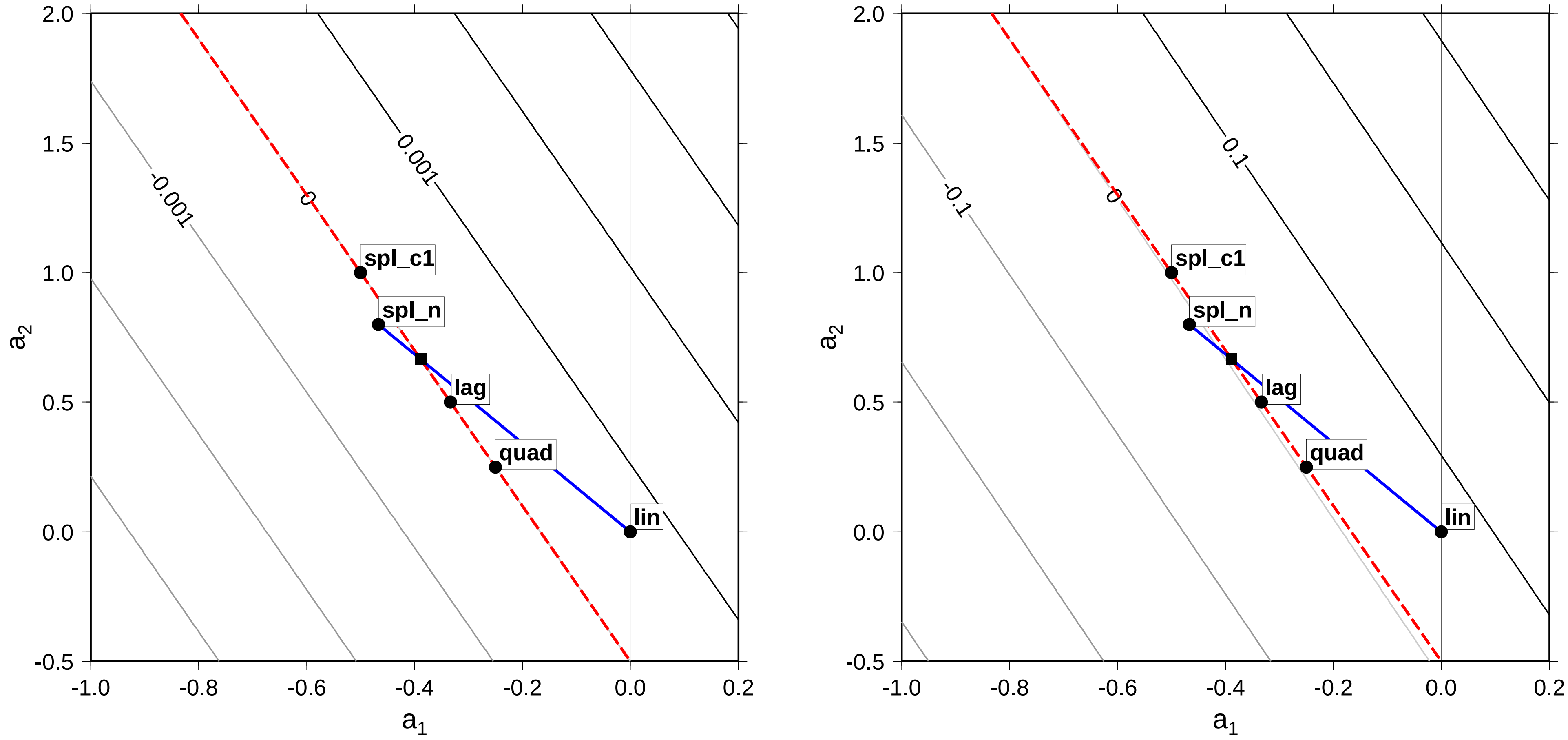
- integration timesteps $\Delta t_i \equiv t_i - t_{i-1}$ initialized randomly so that Courant numbers $c\Delta t_i/\Delta x$ are uniformly distributed in interval $[0, 1)$

- amplitude of numerically advected wave decays as $\exp[-\gamma(k)t]$, which enables to determine interpolator damping rate $\gamma(k)$

Diffusivity maps (1)

dimensionless damping rate

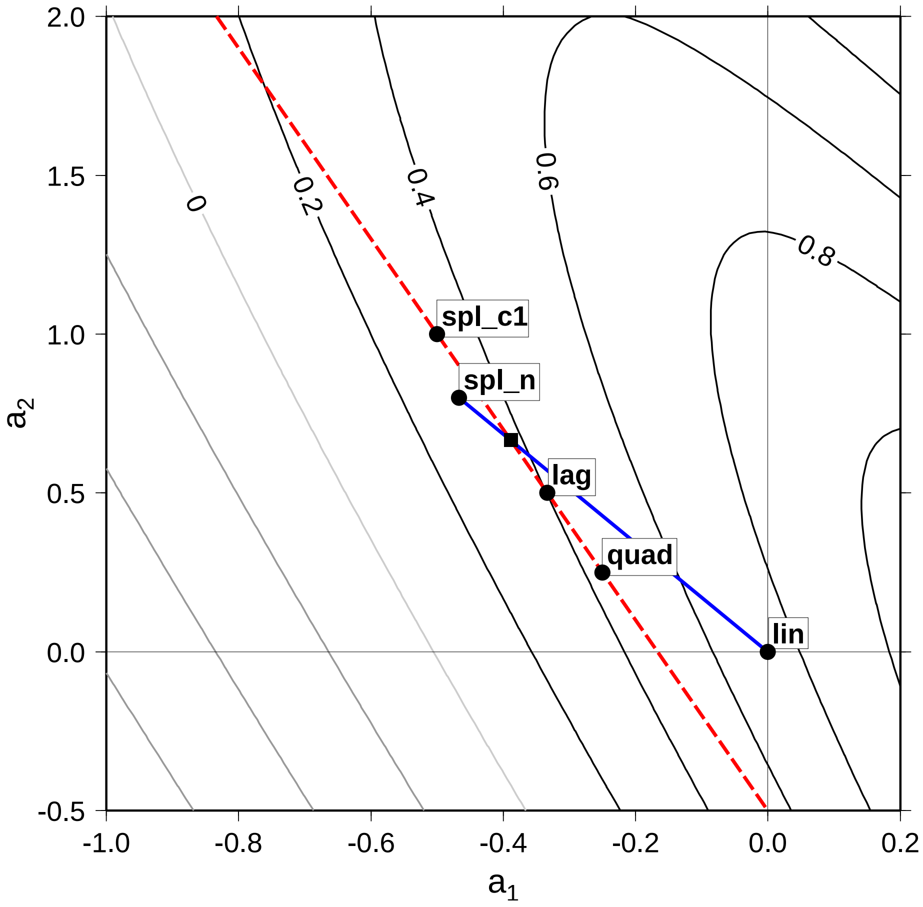
$\lambda = 100\Delta x$ $\lambda = 10\Delta x$



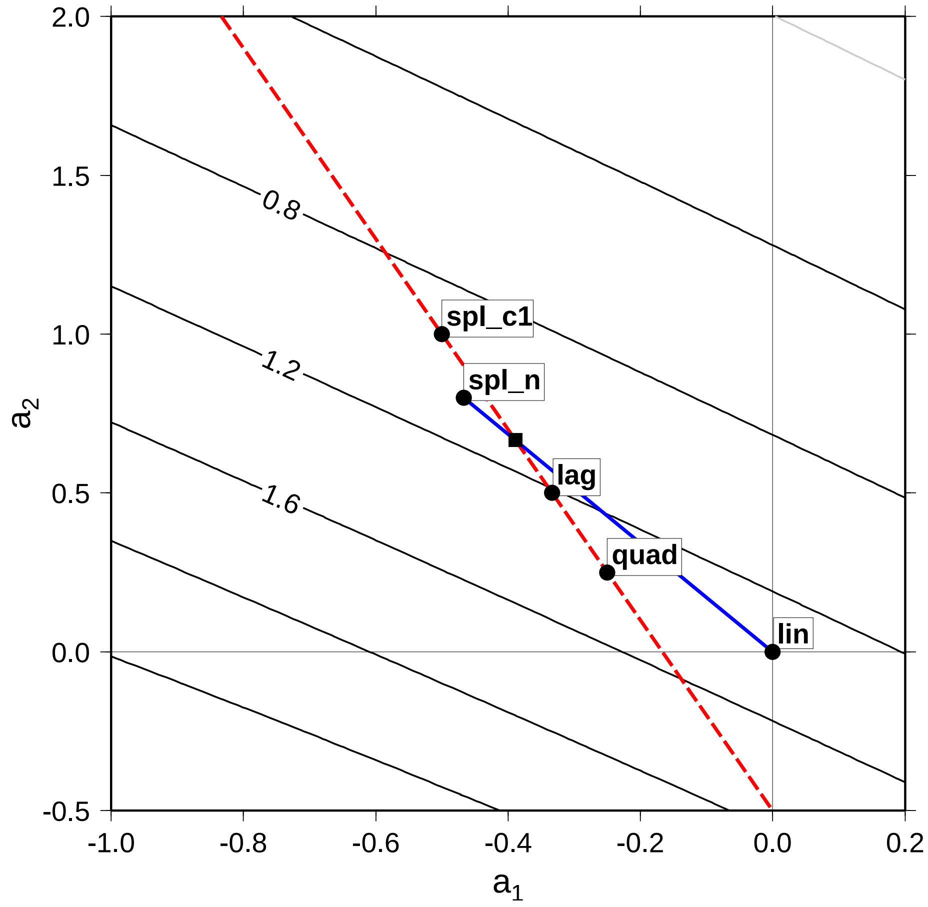
Diffusivity maps (2)

dimensionless damping rate

$\lambda = 3.0\Delta x$



$\lambda = 2.0\Delta x$



Equivalent diffusion

- fitting interpolator damping rate $\gamma(k)$ with power function $D \cdot k^p$ gives estimate for strength D and order p of equivalent diffusion

- fit is good for long waves, where it gives 2nd order equivalent diffusion for linear interpolator and 4th order equivalent diffusion for class of second order accurate interpolators (quadratic, cubic Lagrange polynomial, quasi-cubic spline, . . .)

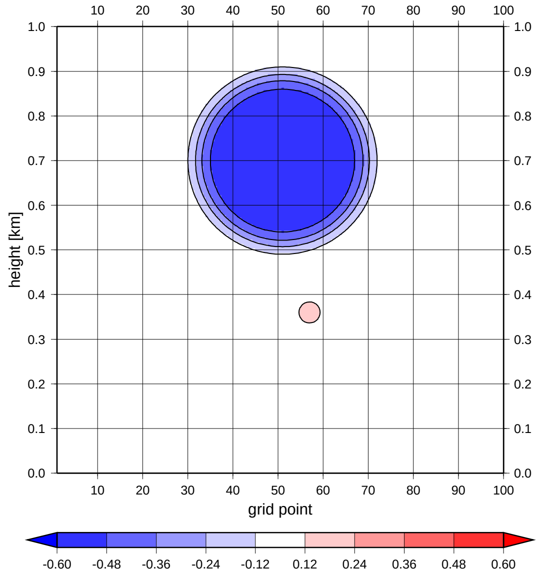
- concept of equivalent diffusion is problematic for natural cubic spline, where $\gamma(k)$ deviates considerably from power law (long waves are amplified, which would mean negative diffusion strength D)

Cold and warm bubble test

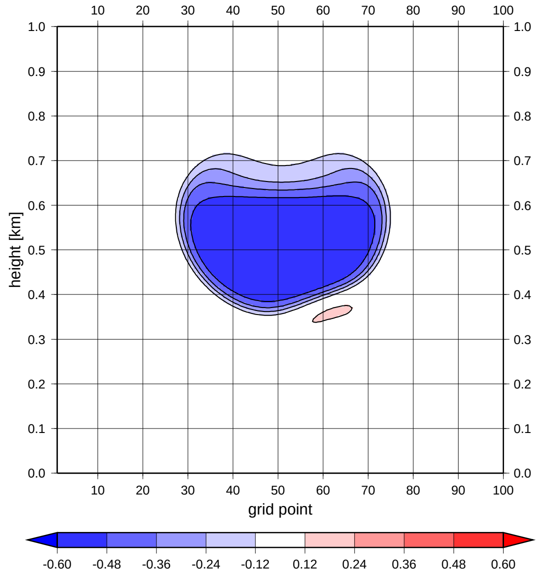
neutrally stratified resting background state in domain 1×1 km, $\Delta x = \Delta z = 10$ m, cold and warm bubble with **smooth edge**

SL2TL ICI scheme with advection of w, $\Delta t = 5$ s

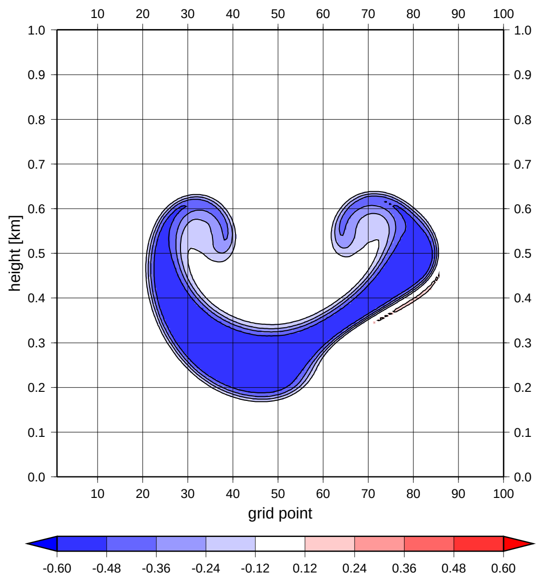
cubic Lagrange interpolator, no other source of damping



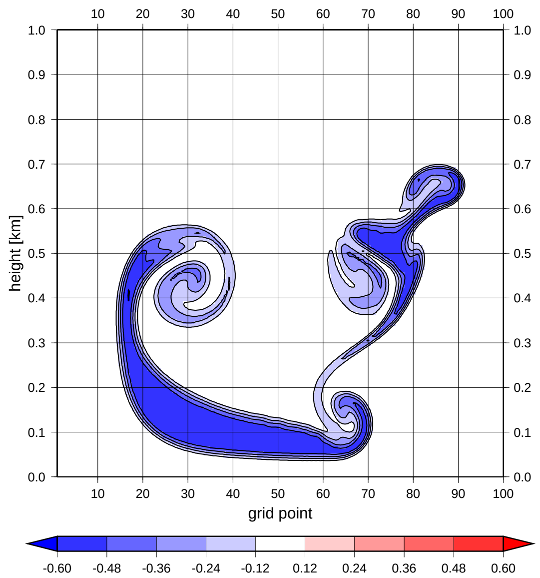
$t = 0$ min



$t = 4$ min



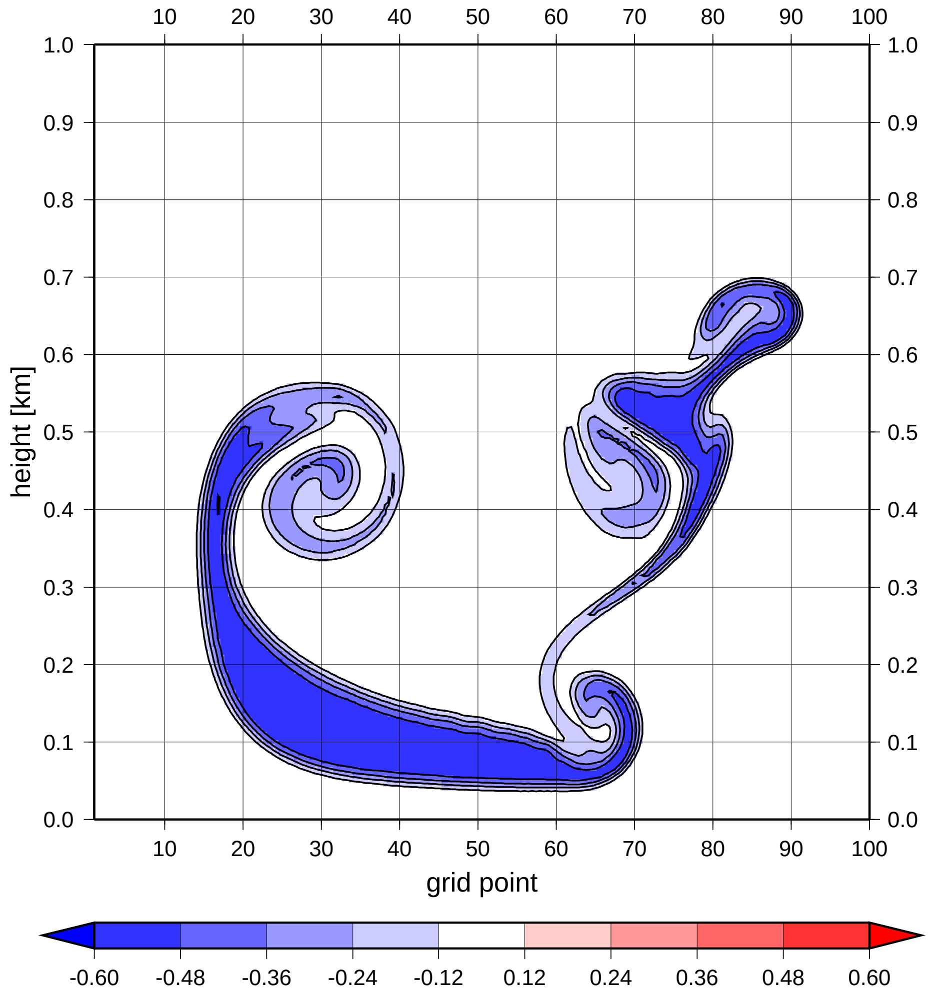
$t = 7$ min



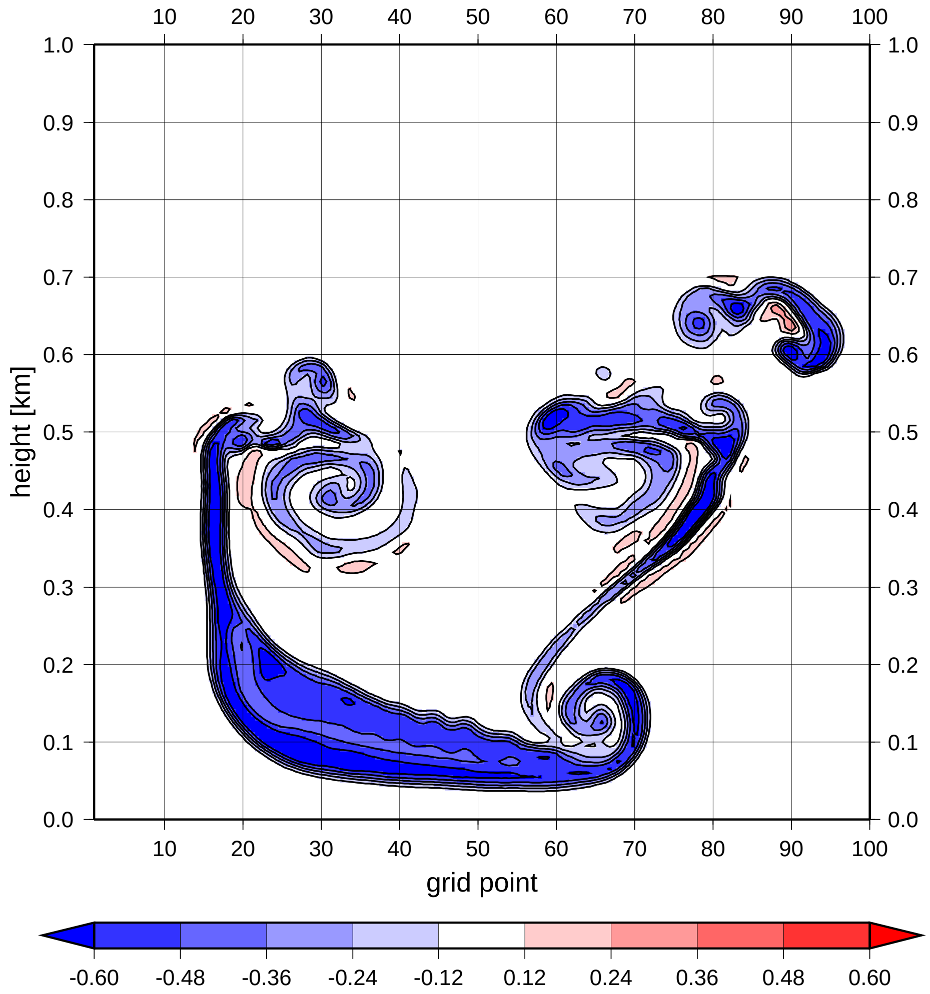
$t = 10$ min

Problem with natural cubic spline (1)

cubic Lagrange polynomial
(reliable reference)

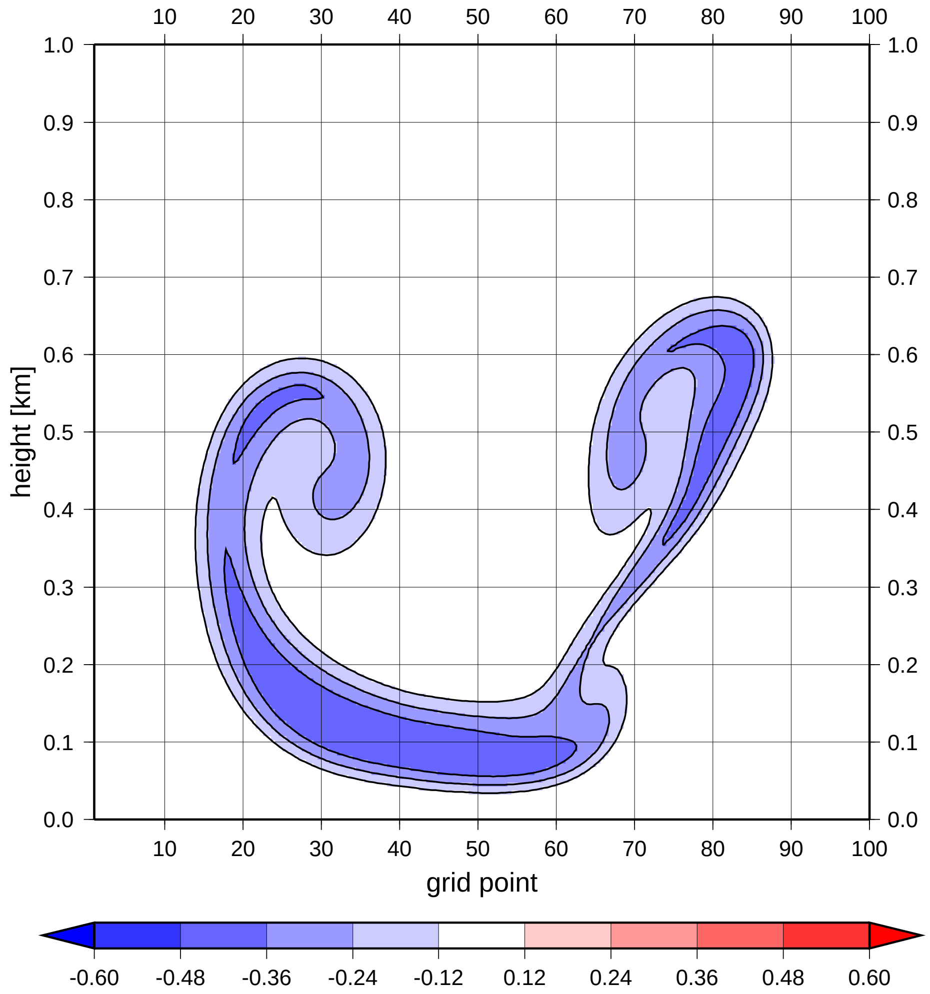


natural cubic spline
(distortion and overshoots)

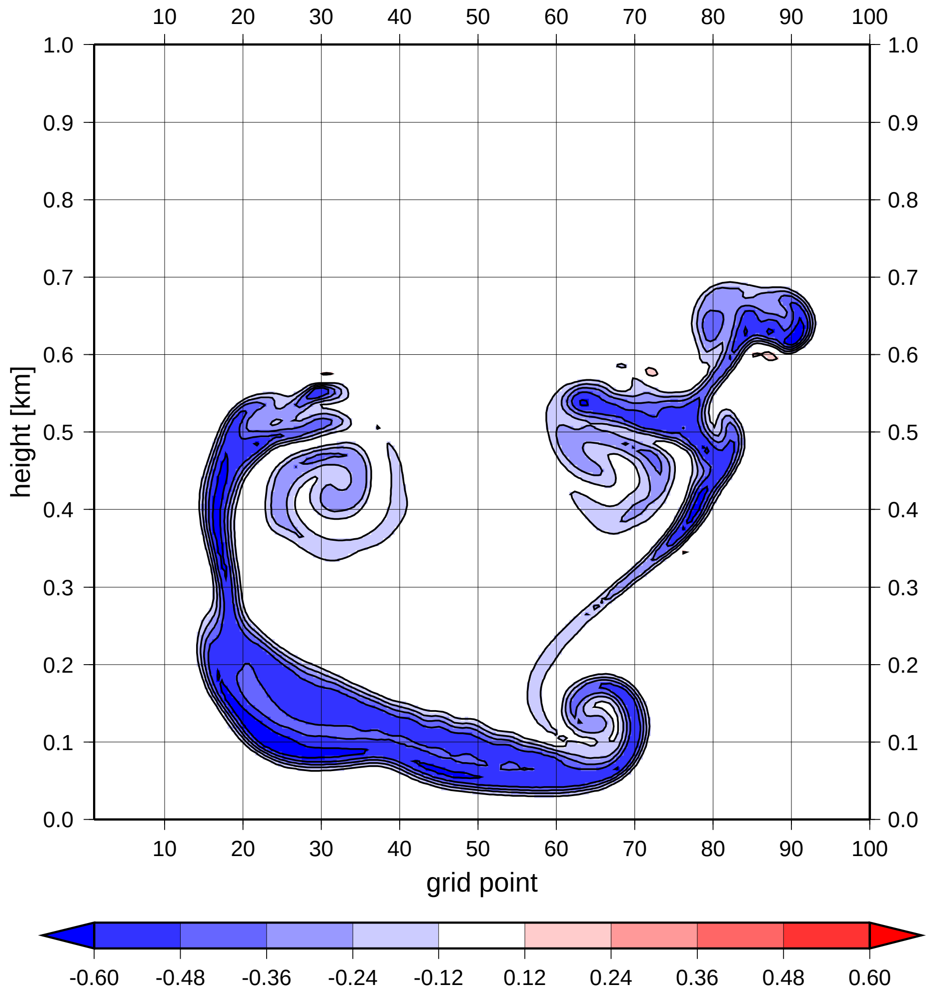


Problem with natural cubic spline (2)

linear interpolator
(details smoothed out)



natural cubic spline, QM version
(distortion; overshoots cut off)



Warm bubble test

- test with **sharp bubble edge** is very sensitive to model resolution

- due to jump at bubble edge numerical solution does not converge as the resolution is increased $\Rightarrow$ it is not good benchmark test

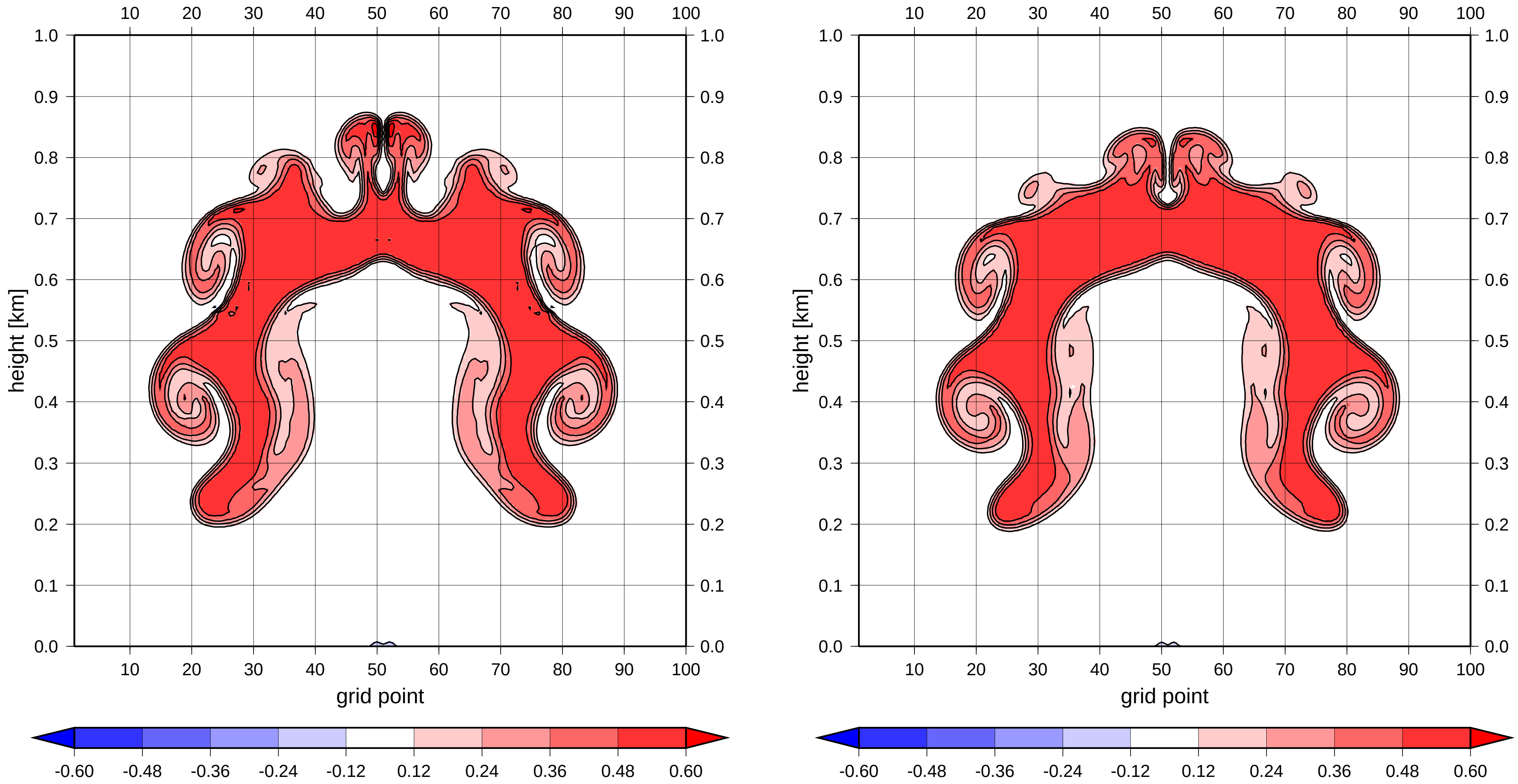
- initial state contains significant proportion of short waves and numerical solution strongly depends on their damping

- it is suitable test for demonstrating damping properties of various interpolators

Damping properties of new interpolators (1)

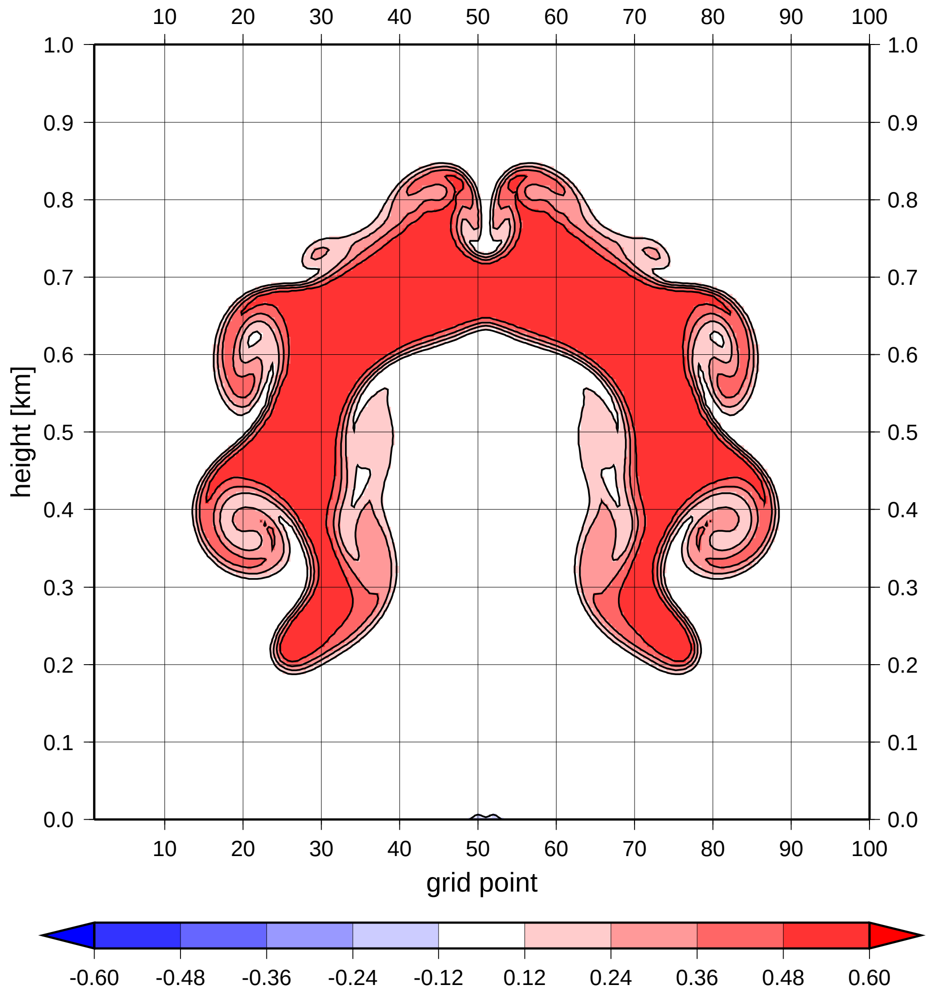
quasi-cubic spline ($\kappa = -2$)

cubic Lagrange polynomial ($\kappa = 0$)

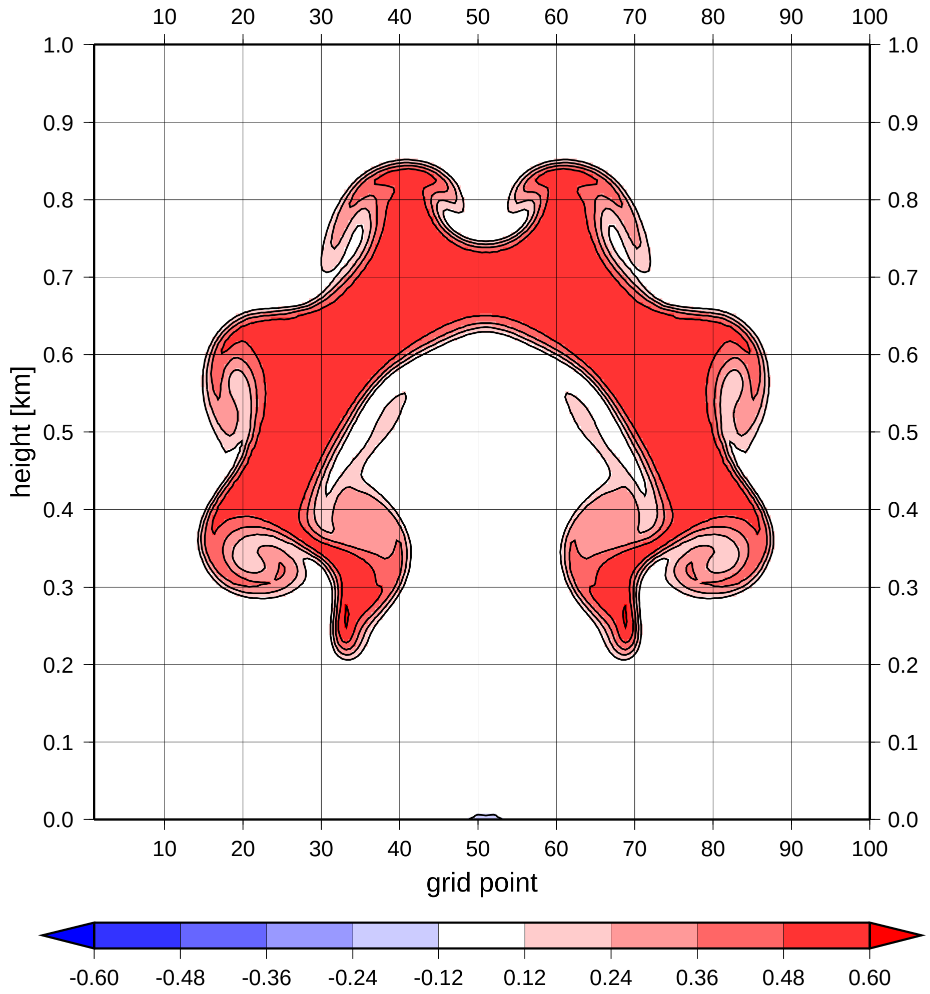


Damping properties of new interpolators (2)

quadratic interpolator ($\kappa = 1$)



strongly diffusive second order accurate interpolator ($\kappa = 6$)



SLHD bonus – Laplacian smoother

- following idea of Pierre Bénard, in the new SLHD scheme Laplacian smoother S can be applied before semi-Lagrangian interpolations:

$$S = 1 + \varepsilon(\Delta x)^2 {\partial_x}^2$$

- using finite difference formula for second order derivative at inner nodes and prescribing it to be zero at outer nodes gives:

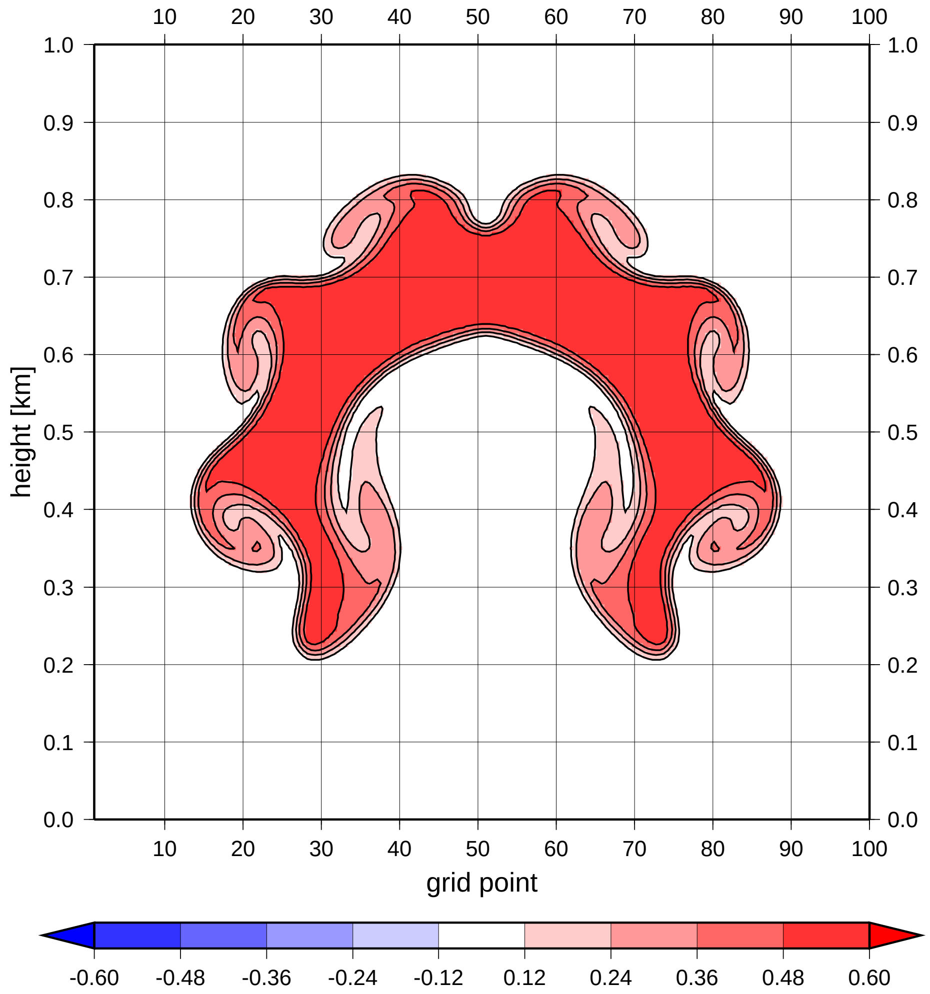
$$S(\mathbf{y}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \varepsilon & 1-2\varepsilon & \varepsilon & 0 \\ 0 & \varepsilon & 1-2\varepsilon & \varepsilon \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_{-1} \\ y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

- high order interpolator F combined with Laplacian smoother S damps more uniformly than diffusive interpolator alone, since it is not much affected by distribution of origin points

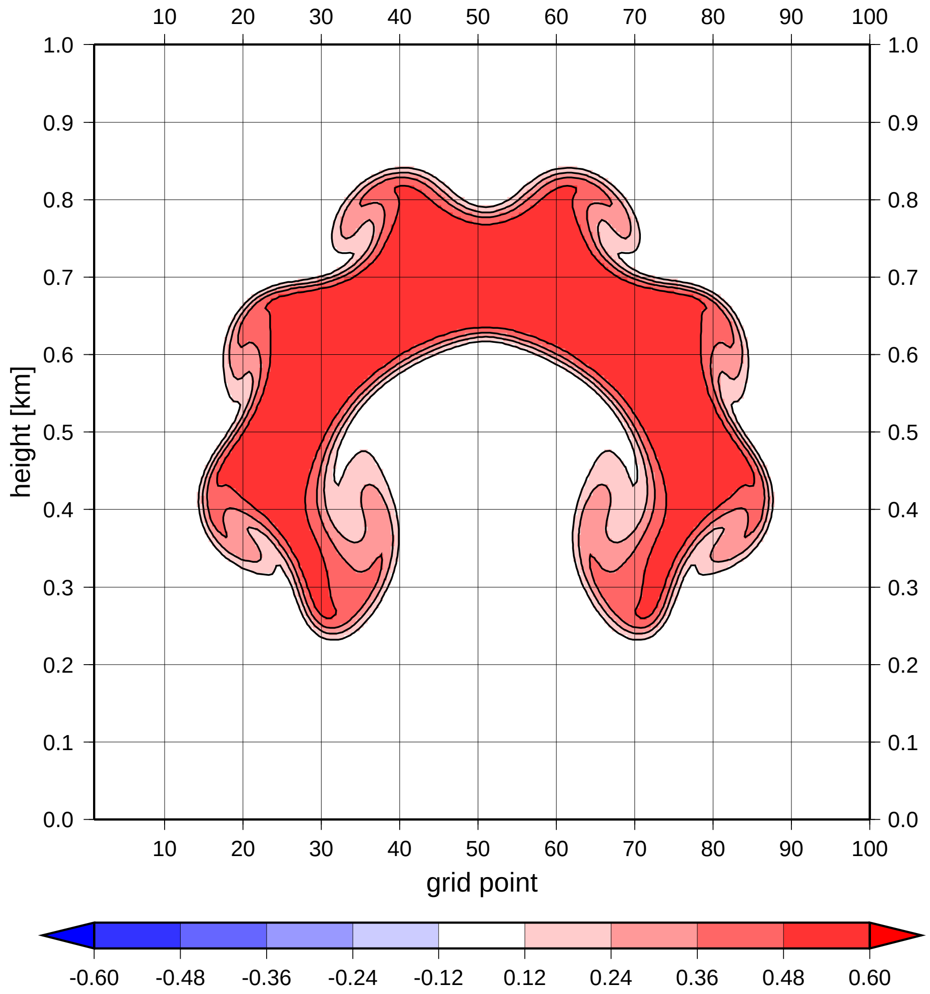
- for stability reasons (explicitly treated diffusion), there is safety limit on smoother strength ε

Damping properties of new interpolators (3)

cubic Lagrange polynomial with smoother, ($\kappa = 0$, $\varepsilon = 0.01$)

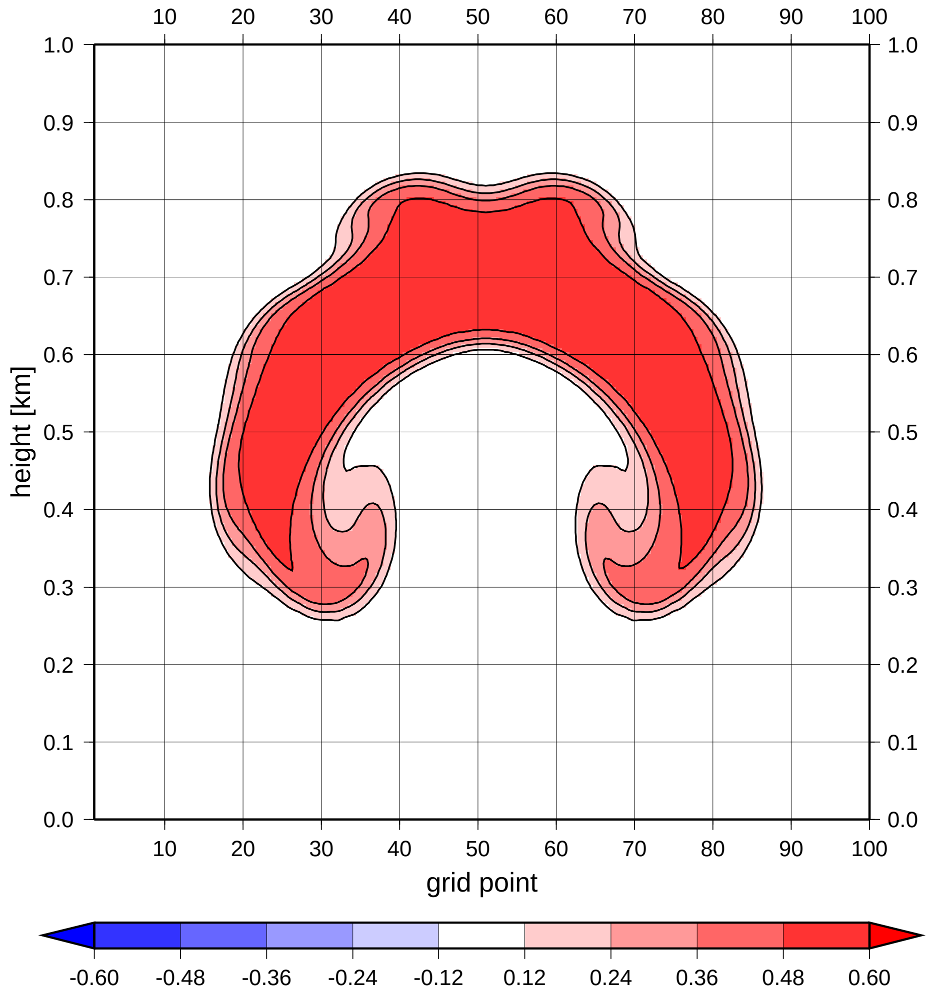


cubic Lagrange polynomial with smoother, ($\kappa = 0$, $\varepsilon = 0.02$)

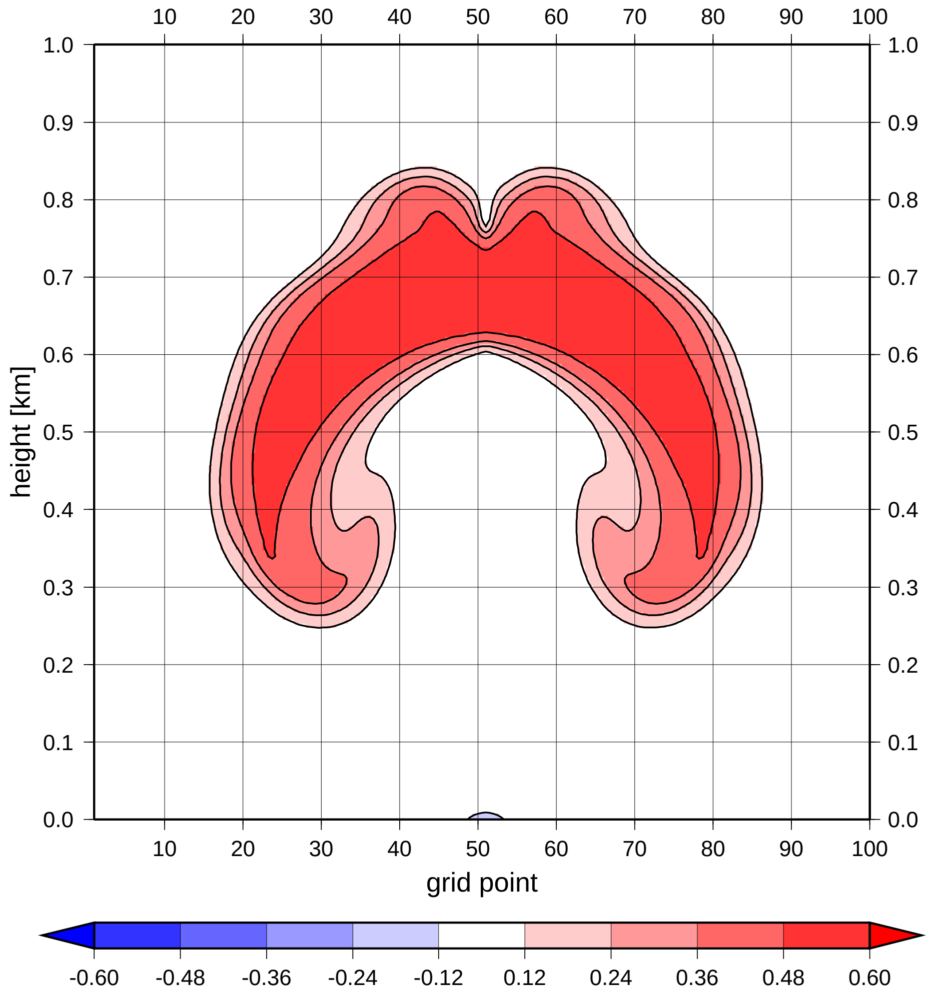


Damping properties of new interpolators (4)

cubic Lagrange polynomial with smoother, ($\kappa = 0$, $\varepsilon = 0.04$)



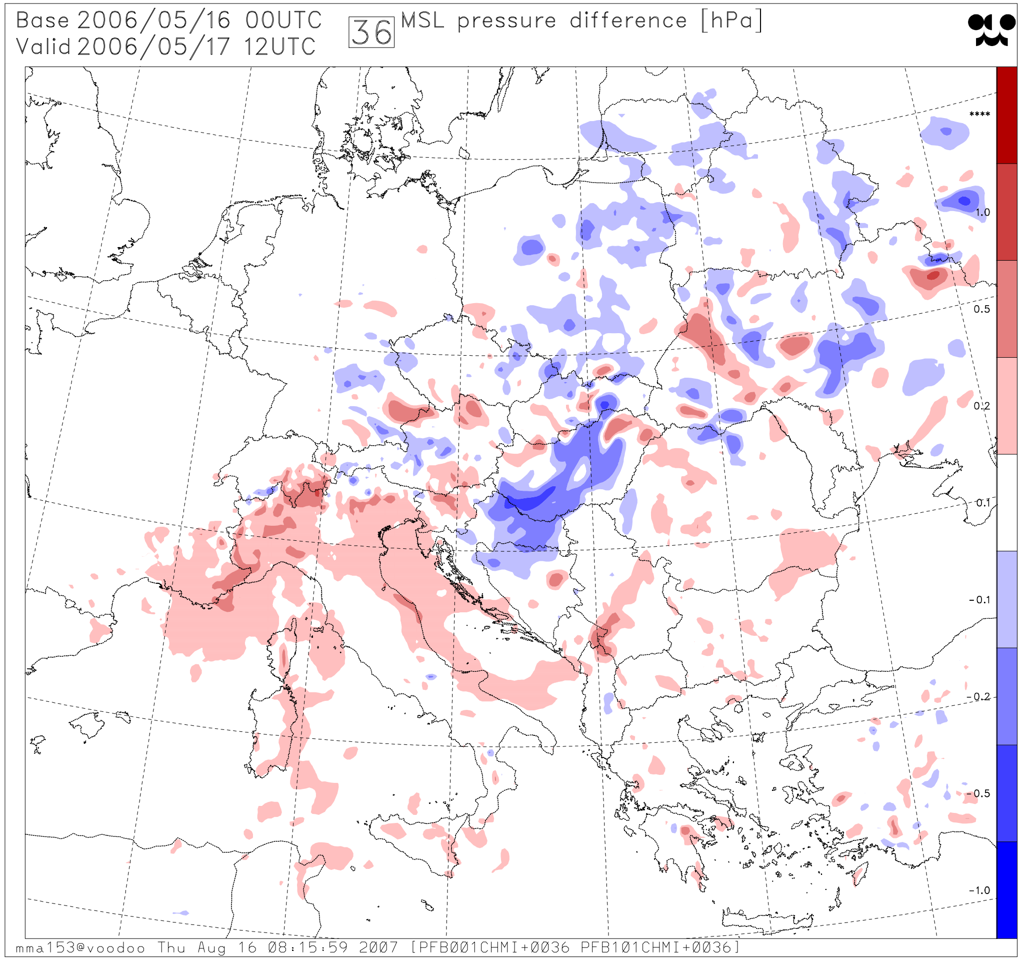
linear interpolator



Mass (non)conservation again

new SLHD scheme, MSLP difference against reference without SLHD

+36 h forecast



Summary

- problems seen in old SLHD scheme led to the proposal of new semi-Lagrangian interpolators with improved accuracy and higher scale selectivity
- new interpolators are second order accurate, equivalent to 4th order diffusion and their damping strength is highly adjustable
- additional degree of freedom is provided by Laplacian smoother which is equivalent to 2nd order diffusion
- new semi-Lagrangian interpolators including TL/AD code were implemented in ARPEGE/ALADIN cycle 35t2 and they will be available in common ARPEGE/IFS cycle 36
- code now reached stable shape, but the full potential of new interpolators was not explored yet (extensive 3D testing is needed)
- use of new semi-Lagrangian interpolators is not restricted to SLHD scheme, they can be used also in static mode