Proposal to include orography in the vertical Laplacian of the linear model used in the ICI time scheme

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The SI time scheme with constant coefficients has usually flat orography in the basic state, but we do not have to stick exactly to linearization around a basic state, we can first linearize, then approximate orographic terms and make some simplifications. This approach was already shown efficient with two temperatures SITR and SITRA where after linearization the reference temperature differs in different terms of the linear model. Similarly in blended hydrostatic and nonhydrostatic systems, several control parameters are introduced in the full model and their value is chosen generally independently in the linear and the full model after linearization.

1 Orography approximation in the linear model

The full Euler equations cast in hybrid η–coordinate write:

$$\frac{dT}{dt} = \frac{\kappa}{\kappa - 1} T(D + d) \tag{1a}$$

$$\frac{d\vec{v}}{dt} = -RT\frac{\nabla\pi}{\pi} - RT\nabla\hat{q} - \nabla\phi - \frac{1}{m}\frac{\partial(p-\pi)}{\partial\eta}\nabla\phi \tag{1b}$$

$$\frac{dgw}{dt} = \frac{g^2}{m}\frac{\partial(p-\pi)}{\partial\eta} \tag{1c}$$

$$\frac{d\hat{q}}{dt} = \frac{1}{\kappa - 1}(D + d) - \frac{\dot{\pi}}{\pi} \tag{1d}$$

$$\frac{\partial q_s}{\partial t} = -\frac{1}{\pi_s}\int_0^1 \nabla\cdot m\vec{v}d\eta, \tag{1e}$$

where

$$d = \frac{p}{mRT}\left(\frac{\partial \vec{v}}{\partial \eta}\nabla\phi - \frac{\partial gw}{\partial \eta}\right) \tag{2a}$$

$$\dot{\pi} = \vec{v}\cdot\nabla\pi - \int_0^\eta \nabla\cdot m\vec{v}d\eta' \tag{2b}$$

$$\phi = \phi_s + \int_\eta^1 \frac{mRT}{p}d\eta'. \tag{2c}$$

We assume linear approximation of pressure perturbation $p - \pi \approx \pi^*\hat{q}$. The linear model writes:

$$\frac{dT}{dt} = \frac{\kappa}{\kappa-1}T^*(D+d) \tag{3a}$$

$$\frac{d\vec{v}}{dt} = -RT^*\frac{\nabla\pi}{\pi^*} - RT^*\nabla\hat{q} - \nabla\phi - \frac{1}{m^*}\frac{\partial\pi^*\hat{q}}{\partial\eta}\left(\nabla\phi\right)^* \tag{3b}$$

$$\frac{dgw}{dt} = \frac{g^2}{m^*}\frac{\partial\pi^*\hat{q}}{\partial\eta} \tag{3c}$$

$$\frac{d\hat{q}}{dt} = \frac{1}{\kappa-1}(D+d) + \frac{1}{\pi^*}\int_0^\eta m^*Dd\eta' \tag{3d}$$

$$\frac{\partial q_s}{\partial t} = -\frac{1}{\pi_s}\int_0^1 m^*Dd\eta. \tag{3e}$$

Then the definition of d can be linearized in

$$d = \frac{\pi^*}{m^*RT^*}\left(\frac{\partial\vec{v}}{\partial\eta}\left(\nabla\phi\right)^* - \frac{\partial gw}{\partial\eta}\right). \tag{4}$$

The idea (of Jozef/Fabrice) is to approximate the linear part of geopotential gradient $(\nabla\phi)^*$ with $\max(||\nabla\phi_s||)S(\eta) = g\Lambda S(\eta)$ and use it in the implicit part of the model. Since the slope Λ may get positive (up hill) or negative values (down hill), it does not make sense to represent its value in the linear model with non zero number. On the other hand, in second order terms in Λ, the sign does not play a role any more and the linear slope could be represented with the maximum value in the domain regardless from the sign. Thus we keep only terms which are of the second order in Λ or which are independent from orography.

The linearized equations, which contain $(\nabla\phi)^*$, can be written as:

$$\begin{aligned}
\frac{\partial\vec{v}}{\partial t} &= -R\mathcal{G}^*\nabla T - RT^*\nabla q_s + RT^*(\mathcal{G}^*-1)\hat{q} - \frac{1}{m^*}\frac{\partial\pi^*\hat{q}}{\partial\eta}g\Lambda S(\eta) \\
\frac{\partial gw}{\partial t} &= g^2\frac{1}{m^*}\frac{\partial\pi^*\hat{q}}{\partial\eta} \\
d &= \frac{\pi^*}{m^*RT^*}\left[\frac{\partial\vec{v}}{\partial\eta}g\Lambda S(\eta) - \frac{\partial gw}{\partial\eta}\right] = \frac{1}{RT^*}\left[g\Lambda S(\eta)\partial^*\vec{v} - \partial^*gw\right].
\end{aligned} \tag{5}$$

We denote $\mathbb{A} = -R\mathcal{G}^*T - RT^*q_s + RT^*(\mathcal{G}^*-1)\hat{q}$ and $\mathbb{B} = \frac{1}{m^*}\frac{\partial\pi^*\hat{q}}{\partial\eta} = (\partial^*+1)\hat{q}$. Then

$$\frac{\partial\vec{v}}{\partial t} = \nabla\mathbb{A} - g\Lambda S\ \mathbb{B} \tag{6}$$

$$\frac{\partial gw}{\partial t} = g^2\mathbb{B}. \tag{7}$$

We apply time differentiation on (5) and substitute with (6) and (7) to get:

$$\begin{aligned}
\frac{\partial d}{\partial t} &= -\frac{1}{RT^*}\left[g^2\partial^*\mathbb{B} - g\Lambda S\partial^*\nabla\mathbb{A} + g^2\Lambda^2 S(S\partial^*\mathbb{B} + (\partial^*S)\,\mathbb{B})\right] \\
&= -\frac{1}{RT^*}\left[g^2(1+\Lambda^2S^2)\partial^*\mathbb{B} - g\Lambda S\partial^*\nabla\mathbb{A} + g^2\Lambda^2\left(S\partial^*S\right)\mathbb{B}\right].
\end{aligned}$$

When omitting first order in Λ term $-g\Lambda S\partial^*\nabla\mathbb{A}$, we come to

$$\frac{\partial d}{\partial t} = -\frac{1}{RT^*}\left[g^2(1+\Lambda^2S^2)\partial^*\left(\partial^*+1\right)\hat{q} + g^2\Lambda^2\left(S\partial^*S\right)\left(\partial^*+1\right)\hat{q}\right) - g\Lambda S\nabla\partial^*\mathbb{A}\right] \tag{8}$$

Since $\mathcal{L}_v^* = \partial^*\left(\partial^*+1\right)$ and all operators of the RHS apply on $\hat{q}$, we can define a new vertical Laplacian as:

$$\mathcal{L}_{new}^* = \alpha\mathcal{L}_v^* + \beta\left(\partial^*+1\right), \tag{9}$$

where $\alpha = \left(1+\Lambda^2S^2\right)$ and $\beta = \Lambda^2\left(S\partial^*S\right)$. For $\Lambda = 0$, $\mathcal{L}_{new}^*$ is equal to $\mathcal{L}_v^*$. We obtain the following prognostic equation for d with changed vertical Laplacian operator and an additional term:

$$\frac{\partial d}{\partial t} = -\frac{g^2}{RT^*}\mathcal{L}_{new}^*\hat{q} + \frac{g\Lambda S\nabla}{RT^*}\partial^*\mathbb{A}. \tag{10}$$

2 Elimination of D in implicit system with slope and discretization hints

For elimination we use the alternative procedure of Fabrice Voitus and eliminate all varibles except D. First, we keep all the orographically induced terms from previous paragraphs. Since $\mathcal{L}_{new}^*$ and the other vertical operators are not commutative we can not eliminate D in the continuous form. We show it is possible in the discrete form. Since there appears a horizontal gradient in γ in discretized equations, we can not separate horizontal and vertical operators. So the equation we get for D will not be a Helmholtz-like equation and we decide to omit the first order in Λ terms everywhere. It means we change only the operator for vertical Laplacian in the linear system from $\mathcal{L}_v^*$ to $\mathcal{L}_{new}^*$.

The definition of $S(\eta)$ is given on full levels and since A and B are defined also on half levels we can defined $S(\eta)$ on half levels with

$$S_{\tilde{l}} = \frac{B_{\tilde{l}}\pi_s^*}{A_{\tilde{l}} + B_{\tilde{l}}\pi_s^*}. \tag{11}$$

similarly to

$$S_l = \frac{B_l\pi_s^*}{A_l + B_l\pi_s^*}. \tag{12}$$

Then we can define $\partial^*S(\eta)$ on full levels as:

$$\begin{aligned}(\partial^*S)_l &= \frac{1}{\delta_l}\left(S_{\tilde{l}} - S_{\widetilde{l-1}}\right) && (13)\\ &= \frac{1}{\delta_l}\frac{\left(A_{\widetilde{l-1}}B_{\tilde{l}} - A_{\tilde{l}}B_{\widetilde{l-1}}\right)\pi_s^*}{\left(A_{\tilde{l}} + B_{\tilde{l}}\pi_s^*\right)\left(A_{\widetilde{l-1}} + B_{\widetilde{l-1}}\pi_s^*\right)} && (14)\end{aligned}$$

In the same way we can define ∂^* on l–th level for other variables:

$$(\partial^*X)_l = \frac{1}{\delta_l}\left((X_{\tilde{l}} - X_{\widetilde{l-1}}\right), \tag{15}$$

but for example, $\hat{q}$ is only defined at full levels, so we need to interpolate it on half levels:

$$X_{\tilde{l}} = \frac{1}{2}\left(X_l + X_{l+1}\right). \tag{16}$$

Then we get

$$(\partial^*X)_l = \frac{X_{l+1} - X_{l-1}}{2\delta_l}, \tag{17}$$

and since $\delta_l \neq \delta_{l+1}$, we replace $2\delta_l$ with $\delta_l + \frac{1}{2}(\delta_{l-1} + \delta_{l+1})$, which is the approximate depth from level $l-1$ to level $l+1$. Thus the discretization of operator $\partial^* + 1$ on level l, for $1 < l < L$, reads

$$((\partial^* + 1)\, X)_l = \frac{X_{l+1} - X_{l-1}}{\delta_l + \frac{1}{2}(\delta_{l-1} + \delta_{l+1})} + X_l \tag{18}$$

and

$$((\partial^* + 1)\, X)_1 = \frac{X_2 - X_1}{\frac{1}{2}(\delta_1 + \delta_2)} + X_1, \tag{19}$$

$$((\partial^* + 1)\, X)_L = \frac{X_L - X_{L-1}}{\frac{1}{2}(\delta_{L-1} + \delta_L)} + X_L \tag{20}$$

on the top and bottom level.

The parameters $\alpha = \left(1 + \Lambda^2 S^2\right)$ and $\beta = \Lambda^2 \left(S\partial^* S\right)$ are constants which multiply $\mathbf{L}_v^*$ and $(\partial^* + 1)$ in the discrete operator $\mathbf{L}_{new}^*$. The discretization of $\mathbf{L}_v^*$ is the same as in the current model.

Eigenvalues

To get the Helmholtz-like problem for D, we need to define the operator $\mathsf{H}^* = I - \delta t^2 \frac{c^2}{H^2} \mathbf{L}_{new}^*$, which need to be invertible and $\mathbf{L}_{new}^*$ should have real and negative eigenvalues (maybe too strong condition). We are testing that in Mathematica (newD.nb) for the Czech operational distribution of 87 levels. Operator H^* is invertible for all $\Lambda \in [0, 5]$ and the operator $\mathbf{L}_{new}^*$ has real negative eigenvalues in all these cases. See Figure 1 for the maximum eigenvalues. Such an operator $\mathbf{L}_{new}^*$ leads to the matrix B, which is invertible and has real positive eigenvalues (must have).

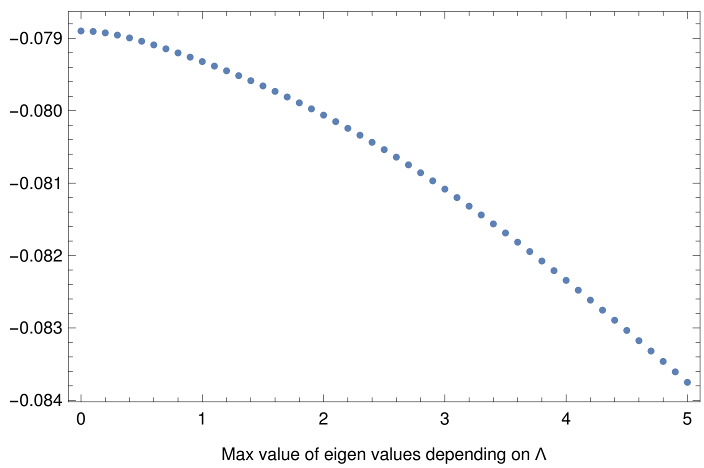


Figure 1: Maximum eigenvalues of $\mathbf{L}_{new}^*$ for $\Lambda \in [0, 5]$.

Code

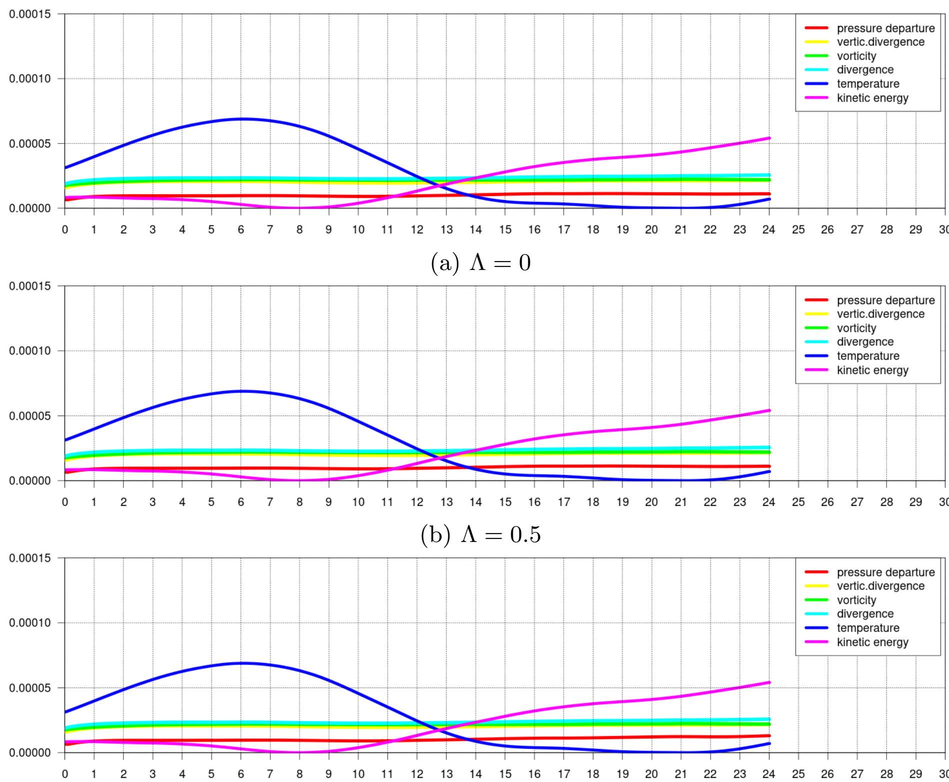
This new vertical Laplacian was implemented in the code based on CY46t1_bf07, available in Prague on kazi1:/local/mma130/CY46t1/CY46t1_bf07. The main change appears in the subroutine SISEVE under the switch L_LINPHI. In subroutine SUNHEEBMAT there is a new logical key to print out the eigenvalues of $\mathbf{L}_{new}^*$. There is a new module YOMDEV, where new switches are declared and where the real parameter LAMBDA_OROG is introduced to control the value of Λ.

3D experiments

We focus on the case of strong winds from 28 October 2017 00 UTC, forecast for 24 hours. We use the Czech operational setting of the model ALARO for 3D simulations (PC scheme with one iteration and SITRA=100K, SLHD, 3MT, ACRANEB2 and TOUCANS, model 2, as physics parameterizations). First, we run simulations on the Czech operational domain (2.325km of horizontal resolution, 87 vertical levels and the time step of 90s) with lateral boundaries provided by Météo France based on ARPEGE. The initial file is provided using 3DVar and CANARI. We test three values of Λ: $\Lambda = 0$, $\Lambda = 0.5$ and $\Lambda = 1$. All experiments are stable with comparable spectral norms as illustrated in Figure 2. Then we run the same setting on the same domain but without time scheme iteration and with the second order extrapolation in time for non-linear terms, i.e. SETTLS scheme. For $\Lambda = 0$, it crashes after 2.4 hours of integration, for $\Lambda = 0.5$ it crashes after almost 3 hours of integration and for $\Lambda = 1$ after 6 hours. As we increase Λ it gets worse.

Thus, we decided to shorten the time step to 75s. With $\Lambda = 0$, the run is still unstable, but already the value of $\Lambda = 0.5$ stabilizes it. We can see that there is a difference in the pressure departure field, mainly in the upper atmosphere. See the evolution of spectral norms for the three values of Λ in Figure 3.

Then we tested the new development in high resolution experiment (200m) over the Alps; see the orography of the domain in Figure 4. We again use all the settings of the Czech operational run (PC scheme), we use lateral boundary conditions from the Czech operational run and the initial file interpolated from the initial file of the Czech operational run and get balanced with DFI. Here, we cut spectral orography with cubic truncation. The time step used is 8s. Here, the usage of $\Lambda = 0$ and $\Lambda = 1$ results in stable runs whose spectral norms evolution is depicted in Figure 5.



(a) $\Lambda = 0$

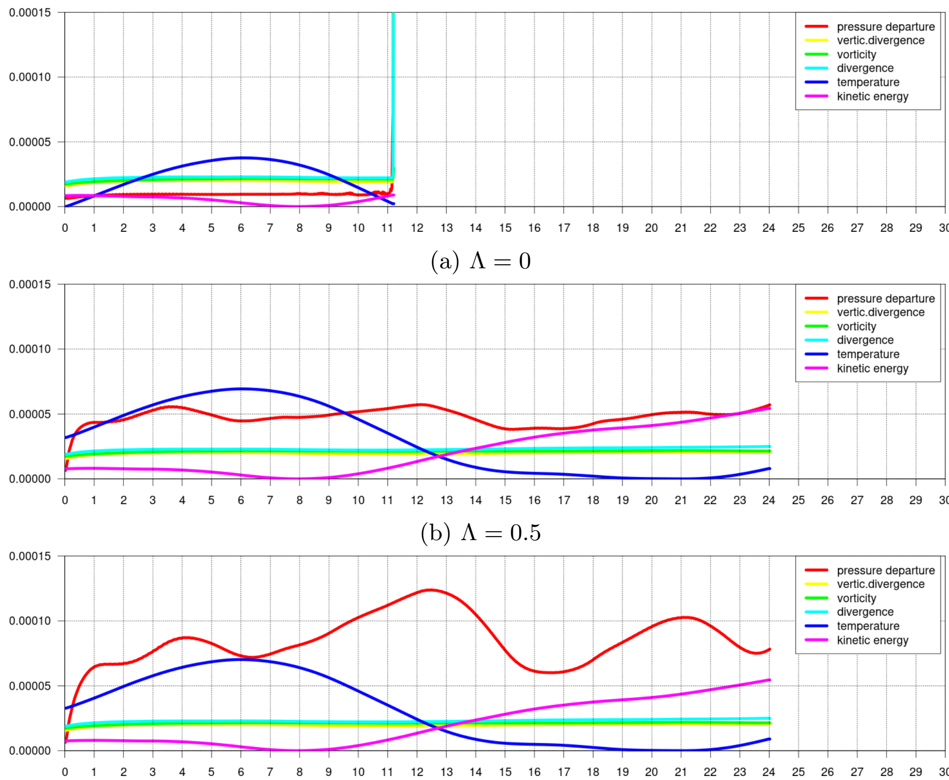
(b) $\Lambda = 0.5$

(c) $\Lambda = 1$

Figure 2: The evolution of model variables spectral norms, PC scheme, 2.3km horizontal resolution, time step = 90 s.

2D experiments

We also test the new Laplacian formulation on 2D experiments. We chose $\Delta z = \Delta x = 200\,m$ and $\Delta t = 15\,s$. The height of orography is given as in Schär et al. [1] and defined by a function $h(x) = \cos^2\left(\frac{\pi x}{\lambda}\right) h_*(x)$, where $h_*(x) = h_0 \cos^2\left(\frac{\pi x}{a}\right)$; for $|x| \leq a$ and 0 otherwise and $h_0 = 2000$m, $\lambda = 8000$m and $a = 10000$m. For such orography the slope is between -0.92772 and 0.92772. For initial horizontal wind v we set the value 4 m/s. It turns out that for $\Lambda = 0$ it runs for 387 steps and the number of steps increases with the increase of Λ to $\Lambda = 0.23$. The results are presented in Table 1 and the propagation of vertical wind w is depicted in Figures 6. Since $\max(||\nabla\Phi_s||) \approx 0.93$, the results in Table 1 indicate



(a) $\Lambda = 0$

(b) $\Lambda = 0.5$

(c) $\Lambda = 1$

Figure 3: The evolution of model variables spectral norms, SETTLS scheme, 2.3km horizontal resolution, time step = 75 s.

that $\Lambda = \frac{1}{g}\max(||\nabla\Phi_s||) = 0.09$ is not the best choice. A bigger value of $\Lambda = 0.23$ is more appropriate here.

Λ	Steps
0	378
0.1	397
0.2	460
0.23	720
0.25	389
0.3	323
0.5	13

Table 1: The number of performed integration steps depending on the Λ.

References

[1] Ch. Schär et al., 2002: *A New Terrain-Following Vertical Coordinate Formulation for Atmospheric Prediction Models*, Month. Weath. Rev. 130, 2459–2480, doi: 10.1175/1520-0493(2002)130<2459:ANTFVC>2.0.CO;2.

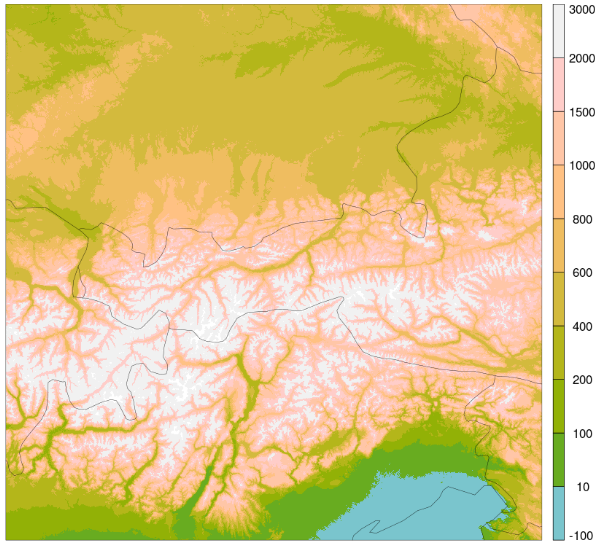
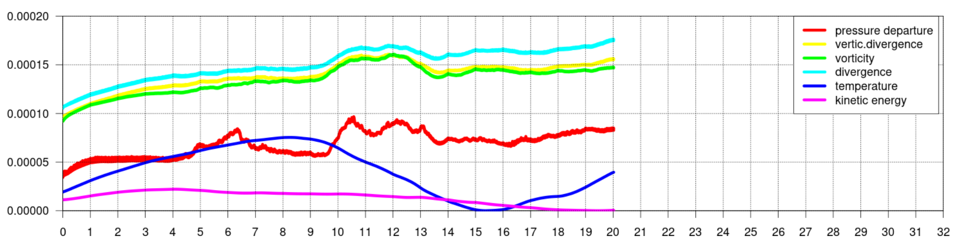
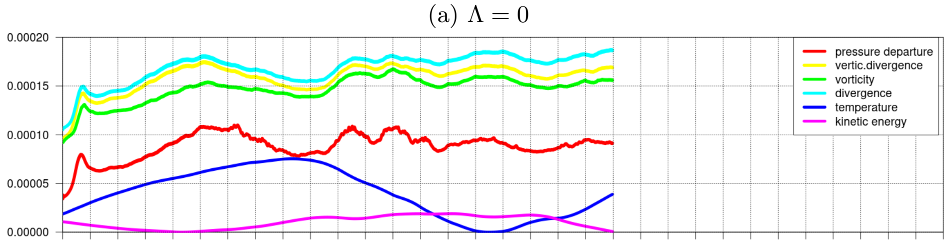


Figure 4: The orography of the 200m experiment.

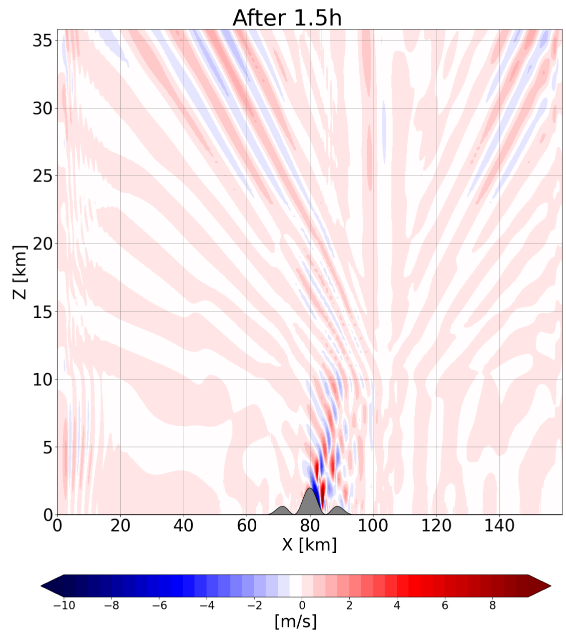


(a) $\Lambda = 0$

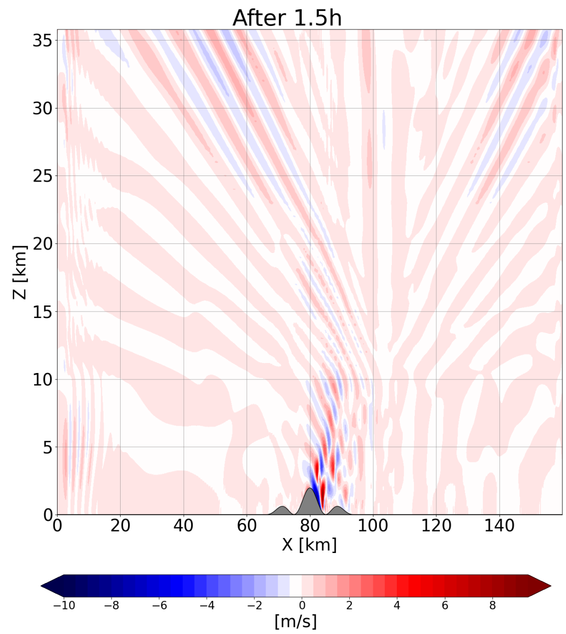


(b) $\Lambda = 1$

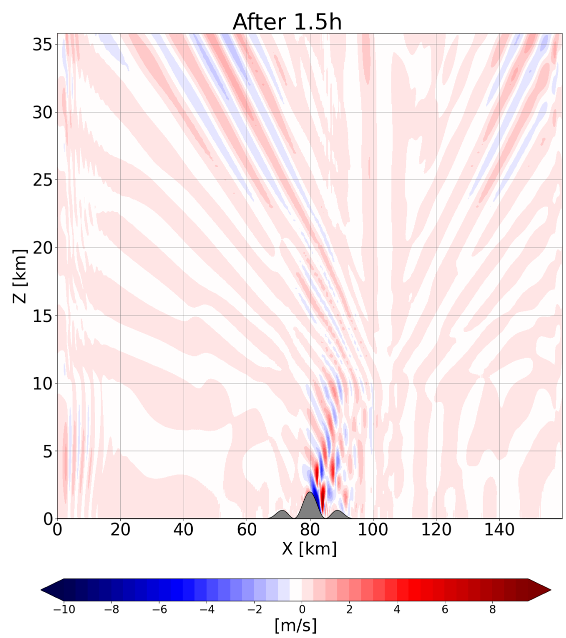
Figure 5: The spectral norms of model variables, PC scheme, horizontal resolution 200m, time step 8 s. It stops after 20 hours because of the time limit.



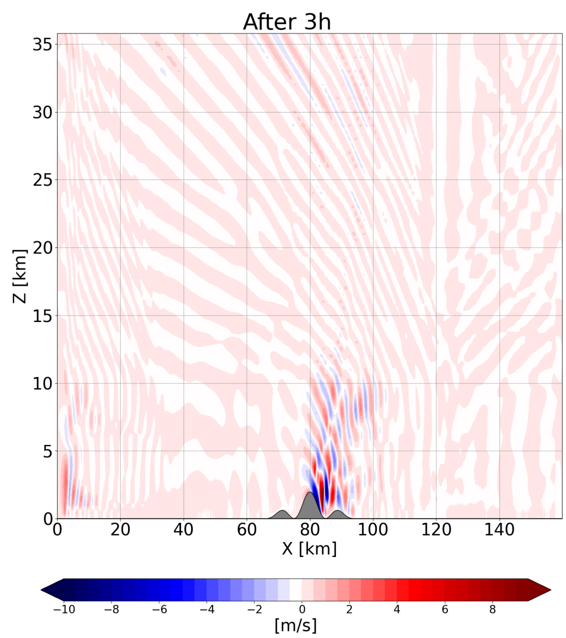
(a) $\Lambda = 0$, after 1.5 h of integration (360 steps)



(b) $\Lambda = 0.1$, after 1.5 h of integration (360 steps)



(c) $\Lambda = 0.23$, after 1.5 h of integration (360 steps)



(d) $\Lambda = 0.23$, after 3 h of integration (720 steps)

Figure 6: Vertical wind in the Schär test.