Stabilization of SI time stepping for ALADIN NH model. Transition of EE system into HY system using set of constant parameters.

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Abstract

Currently HY and NH dynamical cores forms two different worlds in the dynamical core. They represents two approximations to full system of equations. It is possible to reformulate EE system using set of constant parameters ($\alpha, \beta, \gamma, \delta, \epsilon, \tau$) such that solution will form the transition between the two approximations. We call this parameters - NHHY parameters.

When all NHHY parameters are equal to 1 the system is nonhydrostatic, when all parameters are equal to 0, the system is hydrostatic. The trajectory between this two states is not unique. We studied in this report general properties of spectral solver with NHHY parameters. We look at structure equations and wave propagation properties in continuous case.

We studied also stability properties of 2TL SETTLS scheme when looking at stability properties in the spirit of SHB temperature related instability.

We found that there exists a particular set of NHHY parameters that ensures solutions very close to full NH system in troposphere. We verified this result in 2d idealized tests and also in 3d real cases. We stabilize SI SETTLS time stepping for 3d model with resolution 2.3km (CZ operational setting) using NHHY parameters slightly larger than one in linear part of model (over-nonhydrostatic linear model). The nonhydrostatic features in solution are retained as in the reference full NH experiment. We found that stability of SI SETTLS scheme requires consistent settings of horizontal diffusion (diffusion coefficients applied on hor. divergence, vorticity and vert. divergence must be equal, and coefficient applied on temperature and pressure departure must be the same). This was tested with 2nd order diffusion with SLHD settings. When AROME default diffusion is used the 2TL SETTLS scheme is stable as well. AROME uses spectral fourth order diffusion in whole whole domain.

Real 3D weather situations studied in this report represents wind storm case when PC scheme was mandatory for full NH scheme to provide stable forecast. Another case studied, was the case of trapped lee waves which represents purely NH mode in atmosphere. We shows that NHHY transition is capable to capture NH effects and at the same time to ensure stable 2TL SETTLS time stepping. Again in combination with consistent setting of horizontal diffusion.

Further we studied behavior of so called Occitance case. It is case study over south of France with resolution 375m. Domain contains highest part of Alps. PC scheme at such resolution is unstable with 1 iteration. Model doesn't blow but there are spurious oscillations in the middle of troposphere. We run experiment with 3 iterations to obtain stable reference. We stabilize 2TL SI SETTLS time stepping with NHHY parameters, but it requires to suppress NH effect (NHHY parameters 0.3). The solution is similar to reference with 3 iteration, just there are differences in vertical velocities at tropopause level.

We tried to introduce vertically dependent NHHY parameters, but such choice leads to strong instability in model time stepping. This particular possibility remained to be investigated.

Our experiments leads to following conclusions. NHHY approach applied in linear part of model only can stabilize 2TL SL SETTLS scheme at resolutions around 2.3km, keeping full NH solution of nonlinear part. However, at resolutions around 375m also nonlinear part of model have to be modified using NHHY parameters and parameters must be set quite small (0.3). The question is what is between these 2 resolutions. Is it possible for example to stabilize NH model using NHHY approach without significant consequence on model results at 1km resolution ?

I. Methodology

Our aim was to introduce control parameters ($\alpha, \beta, \gamma, \delta, \epsilon, \tau$) in such a way that the solution lays in region between NH and HY solution. The way how to define such parameters is not unique. We require that for $\alpha = \beta = \gamma = \delta = \epsilon = \tau = 0$ the system used will be hydrostatic and for $\alpha = \beta = \gamma = \delta = \epsilon = \tau = 1$ the system used will be EE equations. The mixed state in between is a kind of EE approximation that satisfies limit cases.

We introduce 2 sets of NHHY parameters. One set modifies linear part of model. Such approach can potentially stabilize 2TL SETTLS scheme keeping nonlinear part formulated with full set of EE equations resp. using NHHY parameters in nonlinear part very close to 1. Once we apply NHHY parameters in nonlinear part of model as well, we change character of model solution. As mentioned earlier, there exists multitude of potential trajectories how we can depart from full EE set of equations towards HY set of equations. We investigate how to set NHHY parameters to keep solution with minimal error with respect to reference.

We investigate the stability of SHB kind analyzed with isothermal, motion free, flat terrain context. We require that the stability properties of NESC ICI scheme with NHHY parameters are at least at the level of full NH system. We expected improvement of stability as NHHY approximation converges towards HY system.

We studied also properties of linear continuous system (derived under same assumptions as spectral solver).

II. Nonlinear model

We use prognostic quantities of model ALADIN $\vec{v}, T, q_s = ln(\pi_s), w, \hat{q} = ln(\frac{p}{\pi})$. Here we present the modification of continuous non-linear system in order to achieve required properties.

The vertical and time discretisation of system remained unchanged with respect to existing NH model as described in Benard and Masek documentation.

A. Computation of geopotential

We start with the definition of mass coordinate π in η coordinate (lets call it hydrostatic pressure in the presence of gravity).

$$\frac{\partial \phi}{\partial \eta} = -\frac{mRT}{p}. \tag{1}$$

Mass is in term $m = \frac{\partial \pi}{\partial \eta}$. We divide expression in HY part and NH departure. Then we introduce ϵ parameter that allows us to controls explicitly NH part of relation.

$$\frac{\partial \phi}{\partial \eta} = -\frac{mRT}{p} = -\frac{mRT}{\pi} - \epsilon \left(\frac{\pi}{p} - 1\right) \frac{mRT}{\pi}. \tag{2}$$

with $\frac{p}{\pi} = e^{\hat{q}}$ and $\frac{\pi}{p} - 1 = \frac{1-e^{\hat{q}}}{e^{\hat{q}}}$.

Geopotential is then computed from surface to some arbitrary level η as

$$\phi = \phi_s - \int_\eta^1 \frac{mRT}{\pi} d\eta' - \int_\eta^1 \epsilon \left(\frac{\pi}{p} - 1\right) \frac{mRT}{\pi} d\eta' \tag{3}$$

B. Temperature evolution

Evolution of T is defined in the following way

$$\frac{dT}{dt} = \frac{\kappa T \omega}{\pi} - \alpha \left(\frac{RT}{c_v} D_3 + \frac{\kappa T \omega}{\pi}\right). \tag{4}$$

NH terms are treated as linear departure from HY system.

C. True pressure departure evolution

Evolution of $\hat{q}$ is defined as

$$\frac{d\hat{q}}{dt} = -\delta \left(\frac{c_p}{c_v} D_3 + \frac{\omega}{\pi}\right). \tag{5}$$

In the HY system $\hat{q} = 0$ by definition because $p = \pi$, therefore evolution of $\hat{q}$ is expected to vanish.

D. Horizontal momentum evolution

Horizontal momentum equation in non-rotating frame (coriolis terms not included) in η coordinate gives

$$\begin{aligned} \frac{d\vec{v}}{dt} &= -RT\frac{\vec{\nabla}\pi}{\pi} - \vec{\nabla}\phi \\ &\quad + \beta \left(-RT\vec{\nabla}\hat{q} - \left(\frac{1}{m}\frac{\partial p}{\partial \eta} - 1\right)\vec{\nabla}\phi\right) \end{aligned} \tag{6}$$

We would like to emphasize at this moment that term $\left(\frac{1}{m}\frac{\partial p}{\partial \eta} - 1\right)\vec{\nabla}\phi$ is highly nonlinear ans it has no equivalent terms in current SI operator. There is hidden NH term inside $\vec{\nabla}\phi$ as computation of geopotential itself contains ϵ parameter (see [3]).

E. Vertical momentum evolution

Vertical momentum equation gives

$$\frac{dgw}{dt} = g^2 \frac{1}{m} \frac{\partial \gamma (p - \pi)}{\partial \eta}. \tag{7}$$

The HY system is defined by postulating $\frac{dgw}{dt} = 0$. Therefore in HY system also RHS must be equal to zero.

We use prognostic quantity d in spectral space due to stability reasons. If we introduce γ in the previous way into w equation than the leading term of nonlinear evolution of d yields

$$\frac{dd}{dt} = -g^2 \frac{p}{mRT} \frac{\partial}{\partial \eta} \left[\frac{1}{m} \frac{\partial \gamma (p - \pi)}{\partial \eta} \right] + ... \tag{8}$$

We write only leading vertical laplacian term, which has counterpart in SI system. For complete equation see documentation of NH model from Benard and Masek [2.37].

F. Transformation between gw and d

The open question to be exploited is the treatment of transformation between gw and d variable. It contains true pressure and therefore we could use formulation

$$d = \frac{\pi}{RT} \left[1 + \tau \left(\frac{p}{\pi} - 1 \right) \right] \left(-\frac{1}{m} \frac{\partial gw}{\partial \eta} + \vec{\nabla} \phi \frac{1}{m} . \frac{\partial \vec{v}}{\partial \eta} \right) \tag{9}$$

However, this term has no equivalent term in linear SI system and therefore is not included in our analysis. Term remained to be tested experimentally.

We found that using $\tau = 0$ in full NH experiments with PC scheme has negligible consequences on solution, but part of random instabilities is pacified. There we use setting $\tau = 0$ in all of our experiments presented in this report.

G. Summary of nonlinear system formulation

The system of EE equations is modified by the set of vertically dependent parameters in the following way

$$\begin{aligned}
\frac{\partial q_s}{\partial t} &= -\frac{1}{\pi_s} \int_0^1 \vec{\nabla}(m\vec{v}) d\eta \\
\frac{dT}{dt} &= \frac{\kappa T \omega}{\pi} - \alpha \left(\frac{\kappa T \omega}{\pi} + \frac{RT}{c_v} D_3 \right) \\
\frac{d\vec{v}}{dt} &= -RT \frac{\vec{\nabla}\pi}{\pi} - \vec{\nabla}\phi - \beta \left[RT \vec{\nabla} \hat{q} + \left(\frac{1}{m} \frac{\partial p}{\partial \eta} - 1 \right) \vec{\nabla} \phi \right] \\
\frac{dgw}{dt} &= g^2 \frac{1}{m} \frac{\partial \gamma (p - \pi)}{\partial \eta} \\
\frac{d\hat{q}}{dt} &= -\delta \left(\frac{c_p}{c_v} D_3 + \frac{\omega}{\pi} \right)
\end{aligned} \tag{10}$$

The system is closed with diagnostic relations

$$\omega = \vec{v}\vec{\nabla}\pi - \int_0^\eta \vec{\nabla} (m\vec{v})\, d\eta \tag{11}$$

$$\begin{aligned}
\phi &= \phi_s - \int_\eta^1 \frac{mRT}{\pi} d\eta' \\
&\quad - \int_\eta^1 \epsilon \left(\frac{\pi}{p} - 1 \right) \frac{mRT}{\pi} d\eta'
\end{aligned} \tag{12}$$

$$D_3 = D + d \tag{13}$$

$$d = \frac{\pi}{RT} \left[1 + \tau \left(\frac{p}{\pi} - 1 \right) \right] \left(-\frac{1}{m} \frac{\partial gw}{\partial \eta} + \vec{\nabla} \phi \frac{1}{m} . \frac{\partial \vec{v}}{\partial \eta} \right) \tag{14}$$

When red parameters are set to 0, we obtain hydrostatic set of equations.

III. Linear model and elimination of variables

A. Continuous linear system

After linearisation we obtain following SI linear system. Linear system is formulated with divergence horizontal and vertical instead of wind components form as is done in nonlinear system.

$$
\begin{aligned}
\frac{\partial q_s}{\partial t} &= -\mathbf{N}^* D \\
\frac{\partial T}{\partial t} &= -\kappa T^* \mathbf{S}^* D + \alpha \left[\kappa T^* \mathbf{S}^* D - \frac{RT^*}{C_v}(D+d) \right] \\
\frac{\partial D}{\partial t} &= -R\mathbf{G}^* \nabla^2 T - RT^* \nabla^2 q_s - \nabla^2 \phi_s \\
&\quad + RT^* (\mathbf{G}^* \epsilon - \beta) \nabla^2 \hat{q} \\
\frac{\partial d}{\partial t} &= -\frac{g^2}{RT_a{}^*} \mathbf{L}_v^* \gamma \hat{q} \\
\frac{\partial \hat{q}}{\partial t} &= -\delta \frac{C_p}{C_v}(D+d) + \delta \mathbf{S}^* D.
\end{aligned}
\tag{15}
$$

As mentioned before, parameter τ is not present in the linear system [15] and vertical laplacian $\mathbf{L}_v^*$ is defined as

$$
\mathbf{L}_v^* X = \frac{\pi^*}{m^*} \frac{\partial}{\partial \eta} \left(\frac{\pi^*}{m^*} \frac{\partial}{\partial \eta} + 1 \right) X \tag{16}
$$

Other vertical integral operators $\mathbf{S}^*, \mathbf{G}^*, \mathbf{N}^*$ remained unchanged with respect to existing model formulation.

B. Elimination of time and space discretized system

When system is discretized in time and in vertical direction, we can follow steps reported in Voitus report. Our implicit system to be solved is

$$
\begin{aligned}
q_s - \Delta t \left(-\mathbf{N}^* m^2 D \right) &= \tilde{q}_s \\
T - \Delta t \left\{ -\kappa T^* \mathbf{S}^* m^2 D + \alpha \left[\kappa T^* \mathbf{S}^* m^2 D - \frac{RT^*}{C_v} \left(m^2 D + d \right) \right] \right\} &= \tilde{T} \\
D - \Delta t \left[-R\mathbf{G}^* \nabla^2 T - RT^* \nabla^2 q_s - \nabla^2 \phi_s + RT^* (\mathbf{G}^* \epsilon - \beta) \nabla^2 \hat{q} \right] &= \tilde{D} \\
d - \Delta t \left(-\frac{g}{rH^*} \mathbf{L}_v^* \gamma \hat{q} \right) &= \tilde{d} \\
\hat{q} - \Delta t \left[-\frac{C_p}{C_v} \delta (m^2 D + d) + \delta \mathbf{S}^* m^2 D \right] &= \tilde{\hat{q}}.
\end{aligned}
\tag{17}
$$

We perform elimination of system [17]. We eliminate all variables except horizontal divergence.

and we substitute from [17]. It gives

$$
\begin{aligned}
D - \Delta t^2 \left\{ -R\mathbf{G}^* \nabla^2 \left[-\kappa T^* \mathbf{S}^* m^2 D + \alpha \left(\kappa T^* \mathbf{S}^* m^2 D - \frac{RT^*}{C_v} \left(m^2 D + d \right) \right) \right] \right\} & \\
-\Delta t^2 \left[-RT^* \nabla^2 \left(-\mathbf{N}^* m^2 D \right) \right] & \\
-\Delta t^2 \left\{ +RT^* (\mathbf{G}^* \epsilon - \beta) \nabla^2 \left\{ -\frac{C_p}{C_v} \delta (m^2 D + d) + \delta \mathbf{S}^* m^2 D \right\} \right\} &= D^\bullet
\end{aligned}
\tag{18}
$$

with RHS term being

$$
D^\bullet = \tilde{D} - \Delta t RT^* \nabla^2 \left[(\beta \mathbf{I} - \mathbf{G}^* \epsilon) \tilde{\hat{q}} + \mathbf{G}^* \frac{\tilde{T}}{T^*} + \tilde{q}_s \right]. \tag{19}
$$

after manipulation we obtain

$$
\begin{aligned}
D \quad & -\Delta t^2 \left(gH^* \kappa \mathbf{G}^* \mathbf{S}^* \nabla^2 m^2 D + gH^* \mathbf{N}^* \nabla^2 m^2 D \right) \\
& -\Delta t^2 \left(-gH^* \kappa \mathbf{G}^* \alpha \mathbf{S}^* \nabla^2 m^2 D \right) \\
& -\Delta t^2 \left(+gH^* \frac{R}{C_v} \mathbf{G}^* \alpha \nabla^2 m^2 D + gH^* \frac{R}{C_v} \mathbf{G}^* \alpha \nabla^2 d \right) \\
& -\Delta t^2 \left(-gH^* \frac{C_p}{C_v} \mathbf{G}^* \epsilon \delta \nabla^2 m^2 D + gH^* \frac{C_p}{C_v} \beta \delta \nabla^2 m^2 D \right) \\
& -\Delta t^2 \left(-gH^* \frac{C_p}{C_v} \mathbf{G}^* \epsilon \delta \nabla^2 d + gH^* \frac{C_p}{C_v} \beta \delta \nabla^2 d \right) \\
& -\Delta t^2 \left(+gH^* \mathbf{G}^* \epsilon \delta \mathbf{S}^* \nabla^2 m^2 D - gH^* \beta \delta \mathbf{S}^* \nabla^2 m^2 D \right) \quad = D^\bullet
\end{aligned}
\tag{20}
$$

Terms in the first bracket are HY terms. Additional terms are NH specific terms. When we group terms according variables we obtain

$$
\begin{aligned}
D \quad & -\Delta t^2 \left(gH^* \kappa \mathbf{G}^* \mathbf{S}^* + gH^* \mathbf{N}^*\right) \nabla^2 m^2 D \\
& -\Delta t^2 gH^* \left[\mathbf{G}^* \left(-\tfrac{R}{C_p}\alpha + \epsilon\delta\right) \mathbf{S}^* - \mathbf{G}^* \left(-\tfrac{R}{C_v}\alpha + \tfrac{C_p}{C_v}\epsilon\delta\right) - \beta\delta \mathbf{S}^*\right] \nabla^2 m^2 D \\
& -\Delta t^2 c^{*2} \beta\delta \nabla^2 m^2 D \\
& -\Delta t^2 \left[-gH^* \mathbf{G}^* \left(\tfrac{C_p}{C_v}\epsilon\delta - \tfrac{R}{C_v}\alpha\right) + c^{*2}\beta\delta\right] \nabla^2 d \qquad = \quad D^\bullet
\end{aligned}
\tag{21}
$$

When we introduce constrains

$$
\begin{aligned}
-\tfrac{R}{C_v}\alpha + \tfrac{C_p}{C_v}\epsilon\delta &= \chi \\
\beta\delta &= \zeta
\end{aligned}
\tag{22}
$$

than we see that

$$
-\tfrac{R}{C_p}\alpha + \epsilon\delta = \tfrac{C_v}{C_p}\chi. \tag{23}
$$

Then all parameters in previous equation are replaced by 2 parameters χ and ζ. Taking into account

$$
c^{*2}\tfrac{C_v}{C_p} = gH^* \tag{24}
$$

yields

$$
\begin{aligned}
D \quad & -\Delta t^2 c^{*2} \tfrac{C_v}{C_p} \left(\kappa \mathbf{G}^* \mathbf{S}^* + \mathbf{N}^*\right) \nabla^2 m^2 D \\
& -\Delta t^2 c^{*2} \left[\left(\tfrac{C_v}{C_p}\right)^2 \mathbf{G}^* \chi \mathbf{S}^* - \tfrac{C_v}{C_p}\mathbf{G}^* \chi - \tfrac{C_v}{C_p}\zeta \mathbf{S}^* + \zeta \mathbf{I}\right] \nabla^2 m^2 D \\
& +\Delta t^2 \left[-gH^* \mathbf{G}^* \chi + c^{*2}\zeta\right] \nabla^2 d \qquad = \quad D^\bullet
\end{aligned}
\tag{25}
$$

The same procedure we repeat with equation for vertical divergence

$$
d - \Delta t^2 \tfrac{1}{rH^{*2}} \mathbf{L}_v^* \gamma\delta \left(-gH^* \mathbf{S}^* + c^{*2}\right) m^2 D - \Delta t^2 c^{*2} \tfrac{1}{rH^{*2}} \mathbf{L}_v^* \gamma\delta d = d^\bullet \tag{26}
$$

with RHS term

$$
d^\bullet = \tilde{d} - \Delta t R T^* \frac{1}{rH^{*2}} \mathbf{L}_v^* \gamma \tilde{\hat{q}}. \tag{27}
$$

We introduce new operators which allow more compact form of equations

$$
\begin{aligned}
\mathbf{B}_{HY} &= \tfrac{c_v}{c_p}\left(\kappa \mathbf{G}^* \mathbf{S}^* + \mathbf{N}^*\right). \\
\mathbf{G}_\kappa^* X &= \mathbf{I} - \tfrac{c_v}{c_p}\mathbf{G}^* X \\
\mathbf{S}_\kappa^* X &= \mathbf{I} - \tfrac{c_v}{c_p}\mathbf{S}^* X \\
\mathbf{L}_v^{**} X &= \tfrac{1}{rH^{*2}} \mathbf{L}_v^* X
\end{aligned}
\tag{28}
$$

The equations for D and d than can be written as

$$
\begin{aligned}
\left[\mathbf{I} - \Delta t^2 c^{*2} \left(\mathbf{B}_{HY} + \left[\mathbf{G}_\kappa^* \chi + (\zeta - \chi)\mathbf{I}\right] \mathbf{S}_\kappa^*\right) m^2 \nabla^2\right] D & \\
-\Delta t^2 c^{*2} \left[\mathbf{G}_\kappa^* \chi + (\zeta - \chi)\mathbf{I}\right] \nabla^2 d &= D^\bullet \\
\left[\mathbf{I} - \Delta t^2 c^{*2} \mathbf{L}_v^{**} \gamma\delta\right] d - \Delta t^2 c^{*2} \mathbf{L}_v^{**} \gamma\delta \mathbf{S}_\kappa^* m^2 D &= d^\bullet
\end{aligned}
\tag{29}
$$

We define now $\mathbf{H}_v^*$ as

$$
\mathbf{H}_v^* = \mathbf{I} - \Delta t^2 c^{*2} \mathbf{L}_v^{**} \gamma\delta. \tag{30}
$$

d can be expressed in terms of D as

$$d = \mathbf{H}_v^{*-1}\left[d^\bullet + \Delta t^2 c^{*2}\mathbf{L}_v^{**}\gamma\delta\mathbf{S}_\kappa^* m^2 D\right]. \tag{31}$$

and we can eliminate d from set of equations [29]. We obtain

$$\begin{aligned}\left[\mathbf{I} - \Delta t^2 c^{*2}\left(\mathbf{B}_{HY} + \left[\mathbf{G}_\kappa^*\chi + (\zeta-\chi)\mathbf{I}\right]\mathbf{S}_\kappa^*\right)\nabla^2 m^2\right] D &= D^\bullet + \\ &\quad \Delta t^2 c^{*2}\left[\mathbf{G}_\kappa^*\chi + (\zeta-\chi)\mathbf{I}\right]\nabla^2\mathbf{H}_v^{*-1}\left[d^\bullet + \Delta t^2 c^{*2}\mathbf{L}_v^{**}\gamma\delta\mathbf{S}_\kappa^* m^2 D\right] \\ &= D^{\bullet\bullet} + \Delta t^4 c^{*4}\nabla^2\left[\mathbf{G}_\kappa^*\chi + (\zeta-\chi)\mathbf{I}\right]\mathbf{H}_v^{*-1}\mathbf{L}_v^{**}\gamma\delta\mathbf{S}_\kappa^* m^2 D\end{aligned} \tag{32}$$

with

$$D^{\bullet\bullet} = D^\bullet + \Delta t^2 c^{*2}\nabla^2\left[\mathbf{G}_\kappa^*\chi + (\zeta-\chi)\mathbf{I}\right]\mathbf{H}_v^{*-1}d^\bullet. \tag{33}$$

Resulting Helmholtz equation then yields

$$\left[\mathbf{I} - \Delta t^2\mathbf{B}m^2\nabla^2\right] D = D^{\bullet\bullet} \tag{34}$$

with

$$\mathbf{B} = c^{*2}\left[\mathbf{B}_{HY} + \left[\mathbf{G}_\kappa^*\chi + (\zeta-\chi)\mathbf{I}\right]\mathbf{S}_\kappa^* + \Delta t^2 c^{*2}\left[\mathbf{G}_\kappa^*\chi + (\zeta-\chi)\mathbf{I}\right]\mathbf{H}_v^{*-1}\mathbf{L}_v^{**}\gamma\delta\mathbf{S}_\kappa^*\right] \tag{35}$$

when using definition of $\mathbf{H}_v^*$ [30] we can define vertical laplacian in terms of $\mathbf{H}_v^*$ as $\left(\Delta t^2 c^{*2}\mathbf{L}_v^{**}\gamma\delta\right) = \mathbf{I} - \mathbf{H}_v^*$. Then we finally obtain

$$\mathbf{B} = c^{*2}\left[\mathbf{B}_{HY} + \left[\mathbf{G}_\kappa^*\chi + (\zeta-\chi)\mathbf{I}\right]\mathbf{H}_v^{*-1}\mathbf{S}_\kappa^*\right] = c^{*2}\left[\mathbf{B}_{HY} + \mathbf{B}_{NH}\right]. \tag{36}$$

It as apparent that Helmholtz solver depends finally on three constant parameters χ, ζ and $\gamma\delta$ and Helmholtz solver contains additional term related to NH dynamics

$$\mathbf{B}_{NH} = \left[\mathbf{G}_\kappa^*\chi + (\zeta-\chi)\mathbf{I}\right]\mathbf{H}_v^{*-1}\mathbf{S}_\kappa^*. \tag{37}$$

C. Discussion

We discuss behavior of system with respect to linear SI model.

We see that when we suppress vertical acceleration with $\gamma = 0$ then vertical operator $\mathbf{H}_v^{*-1} = \mathbf{I}$ becomes identity matrix (apparent from definition in [30]), and $\mathbf{B}_{NH} = \left[\mathbf{G}_\kappa^*\chi + (\zeta-\chi)\mathbf{I}\right]\chi\mathbf{H}_v^{*-1}\mathbf{S}_\kappa^*$. $\gamma = 0$ is not sufficient condition for system to become hydrostatic one.

There are 2 ways how the system can become HY one:

- we suppress NH pressure departure evolution with $\delta = 0$. Such choice requires also removal of vertical divergence from T equation $\alpha = 0$. Values of β and ϵ in such case are arbitrary, because they are in terms related to already suppressed NH pressure departure,
- When $\delta > 0$, then we have to ensure that $\zeta = \chi$ and $\chi = 0$. This can be achieved when $\beta = 0$ to remove NH part of pressure gradient force and at the same time $\frac{R}{Cp}\alpha = \epsilon$ must be valid. Such system leads to hydrostatic solution in the context of our study.

D. Structure equation and change of dispersion relations

In this section we very briefly describe elimination of continuous linear system [15] with assumption that all NHHY parameters are constants.

We apply same logical steps as during elimination of Helmholtz solver. Eliminating D and d in space and time continuous case from [15] gives

$$\begin{aligned}\frac{\partial^2 D}{\partial t^2} &= c^{*2}\frac{C_v}{C_p}\left(\kappa\mathbf{G}^*\mathbf{S}^* + \mathbf{N}^*\right)\nabla^2 D \\ &\quad + c^{*2}\left[\left(\chi\frac{C_v}{C_p}\right)^2\mathbf{G}^*\mathbf{S}^* - \chi\frac{C_v}{C_p}\mathbf{G}^* - \zeta\frac{C_v}{C_p}\mathbf{S}^* + \zeta\right]\nabla^2 D \\ &\quad + c^{*2}\left[\zeta - \chi\frac{C_v}{C_p}\mathbf{G}^*\right]\nabla^2 d\end{aligned} \tag{38}$$

and

$$\frac{\partial^2 d}{\partial t^2} = \delta\gamma \mathbf{L}_v^{**} c^{*2}\left(1-\frac{C_v}{C_p}\mathbf{S}^*\right) D + \delta\gamma c^{*2}\mathbf{L}_v^{**} d. \tag{39}$$

We can write

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2} - \delta\gamma c^{*2}\mathbf{L}_v^{**}\right) d &= \delta\gamma \mathbf{L}_v^{**} c^{*2}\mathbf{S}_\kappa^* D \\ \left[\frac{\partial^2}{\partial t^2} - c^{*2}\left(\mathbf{B}_{HY} + [\chi \mathbf{G}_\kappa^* + (\zeta-\chi)]\,\mathbf{S}_\kappa^*\right)\nabla^2\right] D &= c^{*2}\left[\chi \mathbf{G}_\kappa^* + (\zeta-\chi)\right]\nabla^2 d\end{aligned} \tag{40}$$

To make elimination feasible, we assume eigenmodes of above system to have the structure

$$X(x,\sigma) = \hat{X} e^{i\omega t} e^{ikx} \sigma^{i\nu - \frac{1}{2}}. \tag{41}$$

When we assume $\chi = \zeta$ then structure equation is

$$\frac{\partial^4}{\partial t^4}\frac{\partial D}{\partial \sigma} - c^{*2}\frac{\partial^2}{\partial t^2}\left[\gamma\delta\frac{-(\nu^2+\frac{1}{4})}{H^{*2}}\frac{\partial D}{\partial \sigma} - (1-\chi)\frac{N^{*2}H^{*2}}{c^{*2}}\frac{1}{(\nu^2+\frac{1}{4})}\nabla^2\frac{\partial D}{\partial \sigma} - \chi\nabla^2\frac{\partial D}{\partial \sigma}\right] - \gamma\delta c^{*2}N^{*2}\nabla^2\frac{\partial D}{\partial \sigma} = 0. \tag{42}$$

We used $\frac{RC_v}{C_p^2} = \frac{N^{*2}H^{*2}}{c^{*2}}$ and we consider only vertical structure of eigenmodes so far. When time derivatives and horizontal operators are replaced as well and we use substitution $L^2 = \frac{(\nu^2+\frac{1}{4})}{H^{*2}}$ then we obtain dispersion relation

$$\omega^4 - c^{*2}\omega^2\left[\gamma\delta L^2 + (1-\chi)k^2\frac{N^{*2}}{c^{*2}L^2} + \chi k^2\right] + \delta\gamma c^{*2}k^2N^{*2} = 0. \tag{43}$$

We check two limits. Hydrostatic dispersion is retrieved when $\chi = 0$ and $\delta\gamma = 0$. It gives known hydrostatic dispersion relation

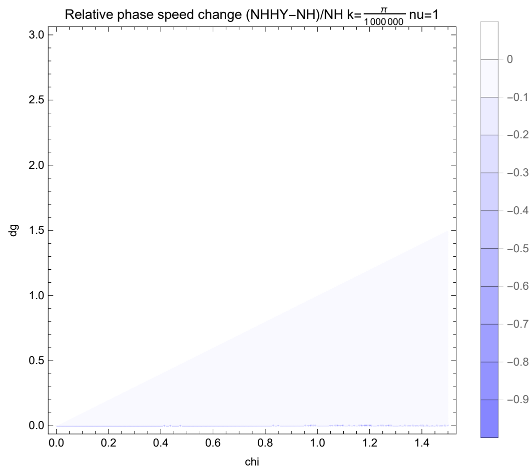
$$\omega^2\left(\omega^2 - k^2\frac{N^{*2}}{L^2}\right) = 0. \tag{44}$$

When $\chi = 1$ and $\delta\gamma = 1$ we obtain nonhydrostatic dispersion relation

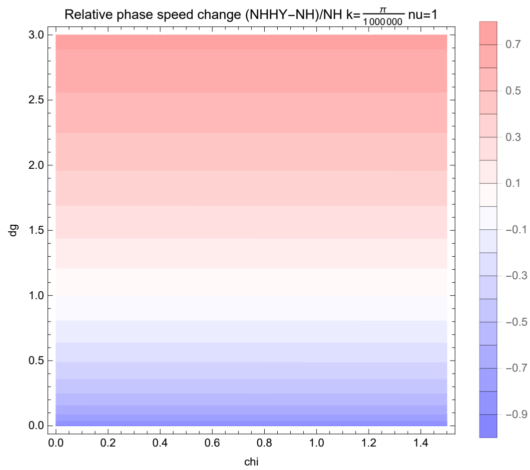
$$\omega^4 - c^{*2}\omega^2\left(L^2 + k^2\right) + c^{*2}N^{*2}k^2 = 0. \tag{45}$$

We analyzed wave properties of horizontally long wave (wavelength $l = 2000km$) and short waves ($l = 2km$). Horizontal wave number is defined as $k = \frac{2\pi}{l}$. We compute relative change of phase speed with respect to full NH solution ($r = \frac{\omega - \omega_{NH}}{\omega_{NH}}$ when ω is solution of (43) and ω_{NH} is solution of (45).

Results for horizontal long wave are on Figure [1]. Here we present the relative change of phase speed with respect to χ (x-axes) and $\delta\gamma$ (y-axes) for vertical mode $\nu = 1$. The properties of long gravity modes remain unchanged. Acoustic modes error is independent of value of χ. The phase speed (frequency) of acoustic modes is increased when $\delta\gamma > 0$ and vice versa.



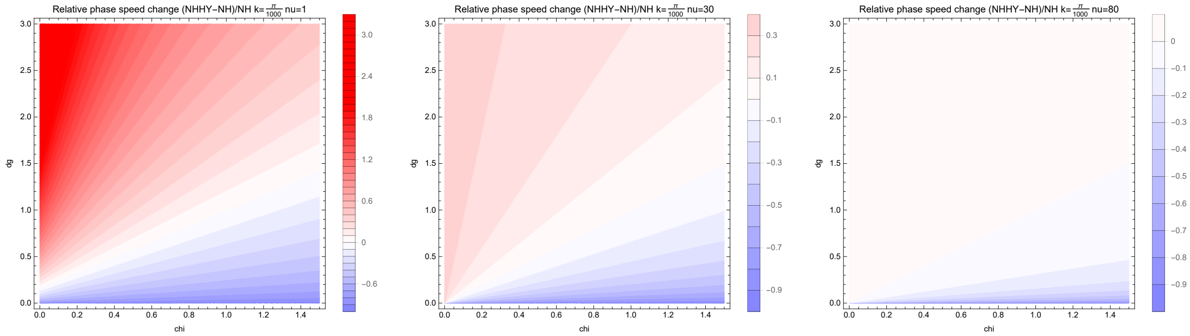
(a) gravity



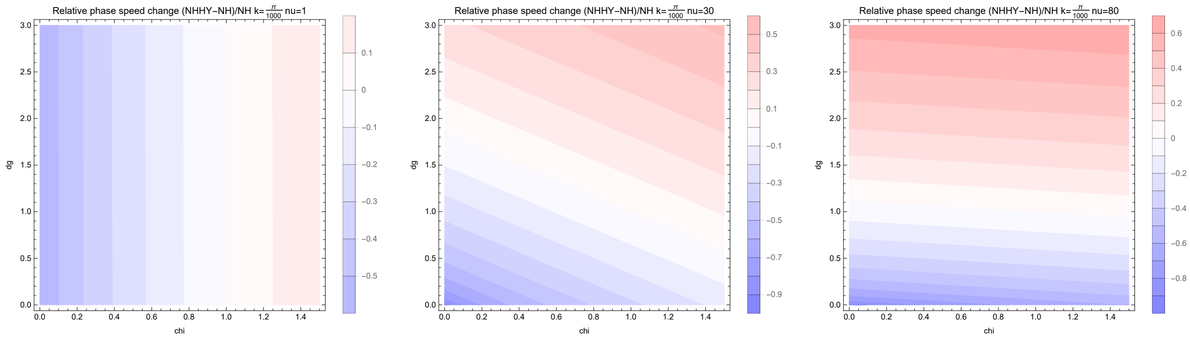
(b) acoustic

Figure 1: Relative change of phase speed $r = \frac{\omega - \omega_{NH}}{\omega_{NH}}$ for modes with $l = 2000km$ and $\nu = 1$. The same results we obtain for any value of ν.

The relative change r for horizontal short waves is shown on Figure [2]. NHHY approximation affects mostly vertical modes with small ν. The short vertical nodes are affected only for very small values of $\delta\gamma$ parameter. Acoustic modes for small ν are independent of $\delta\gamma$ and mode with $\nu = 80$ is independent of χ.



(a) gravity $\nu = 1$ (b) gravity $\nu = 30$ (c) gravity $\nu = 80$



(d) acoustic $\nu = 1$ (e) acoustic $\nu = 30$ (f) acoustic $\nu = 80$

Figure 2: Relative change of phase speed $r = \frac{\omega}{\omega_{NH}}$ for short modes with $l = 2km$ and $\nu = 1$ and $\nu = 10$

In order to look at more detailed behavior of specific choice of NHHY parameters we choose 3 possibilities

1) NHHY1 - $\chi = 1$ and $\delta = 1$ and $\gamma = 3$. This choice is equivalent to choice $\frac{SITR}{SITRA} = 3$.

2) NHHY2 - $\chi = 1.5$ and $\delta = 1.5$ and $\gamma = 1$. This is "over-nonhydrostatic" case,

3) NHHY3 - $\chi = 0.5$ and $\delta = 0.5$ and $\gamma = 1$. Such choice represents approximation of non-linear model with suppressed NH effects. As shown later such choice is needed to stabilize 2TL SI scheme for case with 375m resolution over Alps.

4) NHHY4 - $\chi = 1.5$ and =1.5 and $\gamma = 3$. This represents NHHY case useful for stabilization of SI scheme for resolutions around 2.3km (shown later in this report). This is SITRA combined with NHHY parameters.

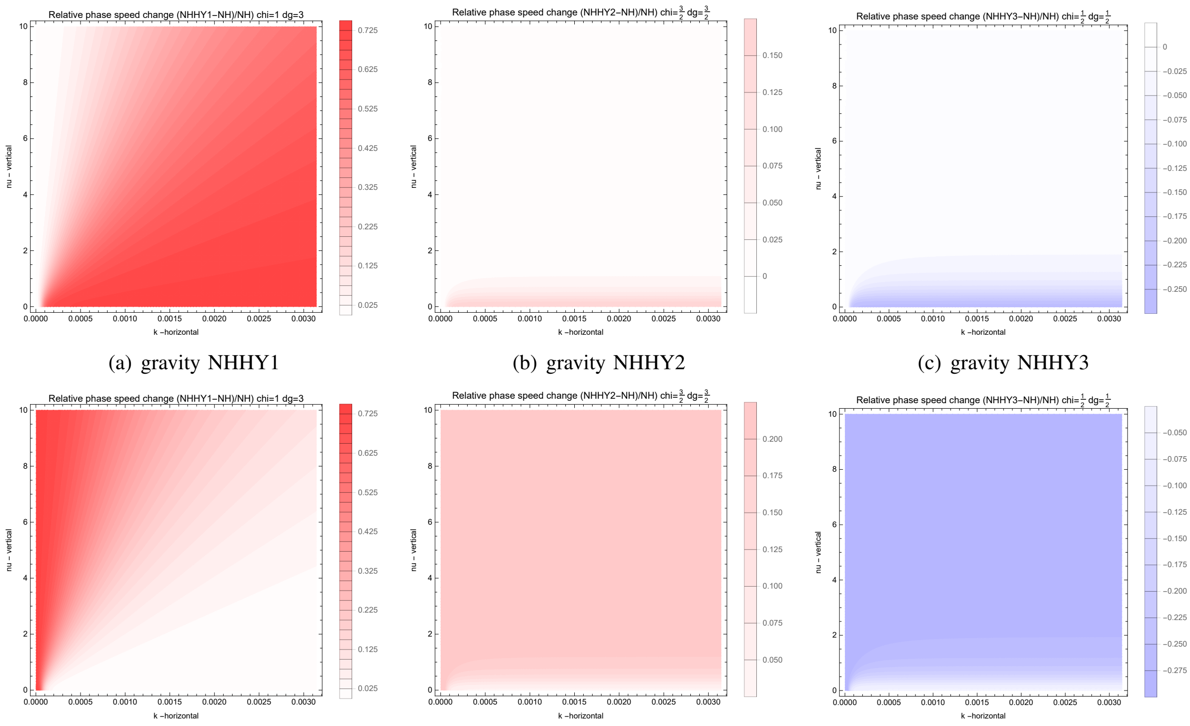
The relative phase speed change is presented on Figure [3] as it depends on horizontal wave number (from $l = 2000km$ to $l = 2km$) and vertical wave number (from $\nu = 0$ to $\nu = 10$). We see that NHHY1 choice increases frequency of gravity modes in whole domain except very long horizontal waves. NHHY2 increases frequency of deep modes ($\nu < 2$) only with the exception of long horizontal waves. NHH3 has the same properties as NHHY2, just frequencies are lowered.

When assuming $\chi = \delta\gamma$ (this is equivalent to assumption $\gamma = 1$) then NHHY approximations affects first vertical modes ($\nu <= 2$) only. These modes are slowed down as parameters becomes smaller than 1 and accelerated as parameter are larger than 1. The properties of high vertical modes ($\nu > 2$) remained unchanged for any horizontal wave number.

The last choice of parameters NHHY4 is compared against NHHY1 choice to filter out effect of γ itself. The results are shown on Figure [4]. It is apparent that relative change of NHHY4 to NHHY1 is the same as relative change between NHHY2 and full NH. Imposing $\chi = \delta > 1$ further increase frequencies on top of $\gamma > 1$. This has stabilization effect as we will show later in this report.

IV. Analysis of stability with continuous system and constant parameters

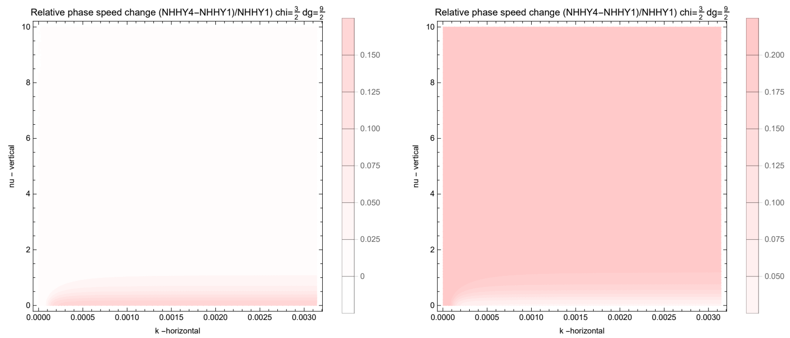
Here we apply procedure documented in Benard, 2003 and Benard at al., 2004. We assume equations are formulated in pure σ coordinate. We assume all parameters to be constant for the purpose of this analysis ($\alpha = \beta = \delta = \epsilon = \gamma = const.$). We



(a) gravity NHHY1 (b) gravity NHHY2 (c) gravity NHHY3

(d) acoustic NHHY1 (e) acoustic NHHY2 (f) acoustic NHHY3

Figure 3: Relative change of phase speed $r = \frac{\omega}{\omega_{NH}}$ for three choices of NHHY parameters. Dependence on horizontal and vertical wave number.



(a) gravity NHHY4 (b) acoustic NHHY4

Figure 4: Relative change of phase speed $r = \frac{\omega}{\omega_{NH}}$ for choice NHHY4. This time the relative change is computed against NHHY1 choice to see effects of NHHY overimposed on top of $\gamma = 3$.

obtain

$$\begin{aligned}
\mathbf{L}_v^* X &= \sigma \frac{\partial}{\partial \sigma}\left(\sigma \frac{\partial}{\partial \sigma} + 1\right) X \\
\mathbf{G}^* X &= \int_\sigma^1 \frac{X}{\sigma'} d\sigma' \\
\mathbf{S}^* X &= \frac{1}{\sigma} \int_0^\sigma X d\sigma' \\
\mathbf{N}^* X &= \int_0^1 X d\sigma'
\end{aligned} \tag{46}$$

In order to get rid of vertical integrals in linear system [15] we use relations

$$\begin{aligned}
\sigma \frac{\partial}{\partial \sigma} \mathbf{G}^* X &= -X \\
\left(\sigma \frac{\partial}{\partial \sigma} + 1\right) \mathbf{S}^* X &= X,
\end{aligned} \tag{47}$$

and we obtain

$$
\begin{aligned}
\sigma\frac{\partial}{\partial\sigma}\frac{\partial D}{\partial t} &= R\nabla^2 T - RT^*\left(\epsilon+\beta\sigma\frac{\partial}{\partial\sigma}\right)\nabla^2\hat{q} \\
\frac{\partial d}{\partial t} &= -\frac{g^2}{RT_a{}^*}\mathbf{L}_v^*\gamma\hat{q} \\
(\sigma\frac{\partial}{\partial\sigma}+1)\frac{\partial T}{\partial t} &= -\kappa T^* D + \alpha\left[\kappa T^* D - \frac{RT^*}{C_v}(\sigma\frac{\partial}{\partial\sigma}+1)\left(D+d\right)\right] \\
(\sigma\frac{\partial}{\partial\sigma}+1)\frac{\partial\hat{q}}{\partial t} &= -\delta\frac{C_p}{C_v}(\sigma\frac{\partial}{\partial\sigma}+1)(D+d)+\delta D.
\end{aligned} \tag{48}
$$

The eigenmodes of system are used as in Benard in the form

$$
X(x,\sigma) = \hat{X}e^{ikx}\sigma^{i\nu-\frac{1}{2}}, \tag{49}
$$

and they are the same for all quantities $D,\hat{q},T,d$. We use relations

$$
\begin{aligned}
\sigma\frac{\partial}{\partial\sigma}X &= \left(i\nu-\tfrac{1}{2}\right)X \\
\left(\sigma\frac{\partial}{\partial\sigma}+1\right)X &= \left(i\nu+\tfrac{1}{2}\right)X \\
\sigma\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial}{\partial\sigma}+1\right)X &= \left(i\nu-\tfrac{1}{2}\right)\left(i\nu+\tfrac{1}{2}\right)X = \left(-\nu^2-\tfrac{1}{4}\right)X \\
\nabla^2 X &= -k^2 X
\end{aligned} \tag{50}
$$

.

Substituting relations we obtain continuous equations that described evolution of time dependent amplitudes $\hat{D},\hat{d},\hat{T},\hat{q}$

$$
\begin{aligned}
\left(i\nu-\tfrac{1}{2}\right)\frac{\partial\hat{D}}{\partial t} &= -k^2R\hat{T}+k^2RT^*\left[\epsilon+\beta\left(i\nu-\tfrac{1}{2}\right)\right]\hat{q} \\
\frac{\partial\hat{d}}{\partial t} &= -\frac{g^2}{RrT^*}\gamma\left(-\nu^2-\tfrac{1}{4}\right)\hat{q} \\
\left(i\nu+\tfrac{1}{2}\right)\frac{\partial\hat{T}}{\partial t} &= -\kappa T^*\hat{D}+\alpha\left[\kappa T^*\hat{D}-\frac{RT^*}{C_v}\left(i\nu+\tfrac{1}{2}\right)\left(\hat{D}+\hat{d}\right)\right] \\
\left(i\nu+\tfrac{1}{2}\right)\frac{\partial\hat{q}}{\partial t} &= -\delta\frac{C_p}{C_v}\left(i\nu+\tfrac{1}{2}\right)(\hat{D}+\hat{d})+\delta\hat{D}.
\end{aligned} \tag{51}
$$

Eigenmodes of system [51] must be purely imaginary in order to maintain stability of time stepping. we write system in matrix form

$$
\begin{pmatrix}
(i\nu-\frac{1}{2}) & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & (i\nu+\frac{1}{2}) & 0 \\
0 & 0 & 0 & (i\nu+\frac{1}{2})
\end{pmatrix}
\begin{pmatrix}
\frac{\partial\hat{D}}{\partial t} \\ \frac{\partial\hat{d}}{\partial t} \\ \frac{\partial\hat{T}}{\partial t} \\ \frac{\partial\hat{q}}{\partial t}
\end{pmatrix} =
$$
$$
\begin{pmatrix}
0 & 0 & -k^2R & k^2RT^*\left[\epsilon+\beta\left(i\nu-\frac{1}{2}\right)\right] \\
0 & 0 & 0 & -\frac{g^2}{RrT^*}\gamma\left(-\nu^2-\frac{1}{4}\right) \\
(\alpha-1)\kappa T^*-\alpha\frac{RT^*}{C_v}\left(i\nu+\frac{1}{2}\right) & -\alpha\frac{RT^*}{C_v}\left(i\nu+\frac{1}{2}\right) & 0 & 0 \\
-\delta\frac{C_p}{C_v}\left(i\nu+\frac{1}{2}\right)+\delta & -\delta\frac{C_p}{C_v}\left(i\nu+\frac{1}{2}\right) & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\hat{D} \\ \hat{d} \\ \hat{T} \\ \hat{q}
\end{pmatrix} \tag{52}
$$

When written in matrix form we could write

$$
\mathbb{L}^*{}_l\frac{\partial X}{\partial t} = \mathbb{L}^*{}_r X \tag{53}
$$

The time evolution of X is given as

$$
\frac{\partial X}{\partial t} = \mathbb{L}^*{}_l^{-1}\mathbb{L}^*{}_r X = \mathbb{L}^* X. \tag{54}
$$

We will analyze the evolution of small amplitude waves in isothermal, resting atmosphere in HY balance with no orography. The evolution of such waves is described by operator $\bar{\mathbb{L}}$ that is the same as $\mathbb{L}^*$ just SI reference temperature T^* is replaced by $\bar{T}=(1+\xi)T^*$ and acoustic temperature parameter $r=1$. SI time stepping then gives

$$
(\mathbf{I}-\tfrac{\Delta t}{2}\mathbb{L}^*)X^{t+\Delta t} = \Delta t(\bar{\mathbb{L}}-\mathbb{L}^*)X^{t+\frac{\Delta t}{2}}+(\mathbf{I}+\tfrac{\Delta t}{2}\mathbb{L}^*)X^t. \tag{55}
$$

We assume $X^{t+\Delta t} = \lambda X^t$ and $X^{t+\Delta t} = \lambda^2 X^{t-\Delta t}$, with λ being complex factor. Stability of time stepping requires $|\lambda| <= 1$ for any value of studied internal parameters.

First analysis is dedicated to stability with zero residual $\xi = 0$. Time stepping is reduced to

$$(\mathbf{I} - \tfrac{\Delta t}{2}\mathbb{L}^*)X^{t+\Delta t} = (\mathbf{I} + \tfrac{\Delta t}{2}\mathbb{L}^*)X^t, \tag{56}$$

and must be unconditionally stable $|\lambda| = 1$ for all values of internal parameters of interest. We perform analysis of stability with following fixed parameters

$$\begin{aligned} k &= \tfrac{\pi}{100.} m^{-1} \\ \nu &= 1 \\ \Delta t &= 1s \\ T^* &= 300K \\ r &= \tfrac{50}{T^*} \end{aligned} \tag{57}$$

The time stepping is unconditionally stable for any choice of parameters when they satisfy constrain [22]. The stability is independent on the choice of γ.

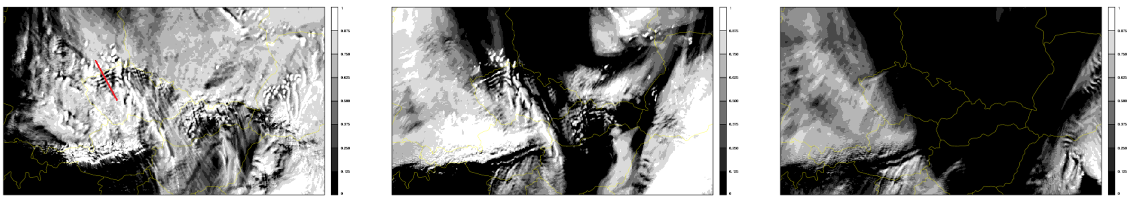
We implemented general analysis of stability during stay. We found bug in implementation at the end of stay, and therefore we have to recompute all results and we can not presents them in the report for the time being.

V. 3D EXPERIMENTS

A. *Case study - trapped lee wave*

The accuracy of the scheme was tested using case from 12th of February 2019 00UTC when NH mountain wave behind Krusne Hory mountain appeared. This wave represents trapped lee wave mode that only NH models are capable to simulate. Simulations are performed using input fields from CHMI operational suite at resolution 2.3km.

The effect was profound at low cloudiness field shown at Figure [**??**]



(a) W001 - PC NH scheme

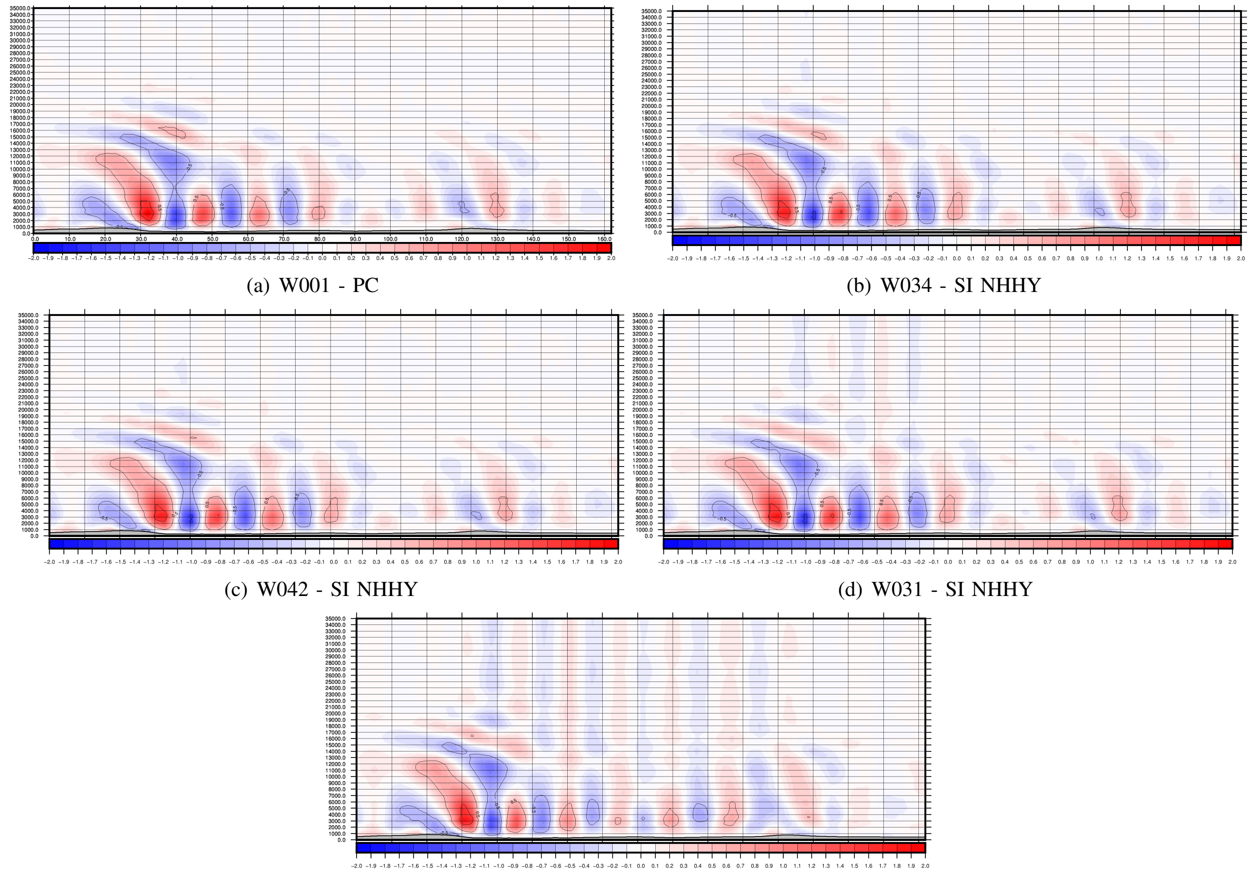
Figure 5: Low, middle and high cloudiness fields. Forecast from 12th of February 2019 00UTC +11h.

All experiments are performed with the same resolution. The trapped lee waves are shown along cross section via Krusne Hory (red line on Figure (5)). Results are shown always at +11h forecast. The reference experiment is obtained using setting of CHMI operational suite. The setting for each experiment are listed in Table [I].

The parameter of diffusion RDAMPPD was set same as RDAMPT. The diffusion parameters RDAMPVD and RDAMPVDS must be set consistent with RDAMPDIV and RDAMPVOR. This setting was necessary condition but not sufficient for stabilization of SI scheme.

When NHHY scheme is activated using LNHHY key, than NHHY parameters are computed for VERSION=1 in following way. Parameter RNHHY_VAL represents some auxiliary quantity h. The NHHY parameters in NL part of model are computed as $\chi = \alpha = h^2$, $\delta = h^k$ (k is represented by RNHHY_DELTA_FAC in namelist), $\beta = \epsilon = h^(2-k)$. Finally $\gamma = h^m$ (m is represented by RNHHY_GAMA_FAC in namelist). Parameters in LI model are defined in the same way, just RNHHY_VALLI represents h in namelist relevant for linear NHHY parameters. Parameter τ controls NH effects in transformation between gw and d. It is represented by by RNHHY_TAU_NL in namelist.

We report results from 3 experiments with NHHY set of parameters. We were able to stabilize SI SETTLS time stepping with all three version of NHHY and the results after 11h of integration are shown on Figure [6]. W034 is experiment with full NH model with modified linear SI operator only. Experiment W042 is experiment when both NL and LI model are modified. The experiment W031 is different in the sense that stabilization of SI SETTLS time stepping was achieved via suppression of NH effect in full model ($h = 0.7$), while linear part of model remained unchanged. We see that in such case there are artifacts of vertical velocity in stratosphere. When we suppress NH effect even more than artifacts becomes even stronger (W027) and for SI SETTLS scheme to be stable we used same NHHY parameters in NL and LI part of model ($h = 0.5$ in both parts).



(a) W001 - PC

(b) W034 - SI NHHY

(c) W042 - SI NHHY

(d) W031 - SI NHHY

(e) W027 - SI NHHY

Figure 6: True vertical velocity w [m/s] cross section. Forecast from 12th of February 2019 00UTC +11h] and text

In order to better understand how NHHY approximation affects stationary mountain waves we performed set of idealized experiments with trapped lee wave case of Keller, 1993.

The results of idealized flow over Agnesi mountain is presented on Figure [7]. Results are always obtain with the same value of χ parameter in both part of model (NL and LI). The reference result is obtained with $\chi = 1$ (full NH model). We see that wave length of solution is not affected in such case. The value of $\chi > 0.4$ modifies solution very slightly. When $\chi = 0.2$ we see significant change in solution as new wave train develops. Therefore we can conclude that for χ close to 1 the solution is not change significantly (more precise statistics remained to be done).

B. Case study - windstorm

To test stabilization of the scheme in realistic 3D context we have chosen the case from 30th of October 2017 from 00UTC. The windstorm over Central Europe caused NH model to be unstable with 2TL SI SETTLS time marching scheme.

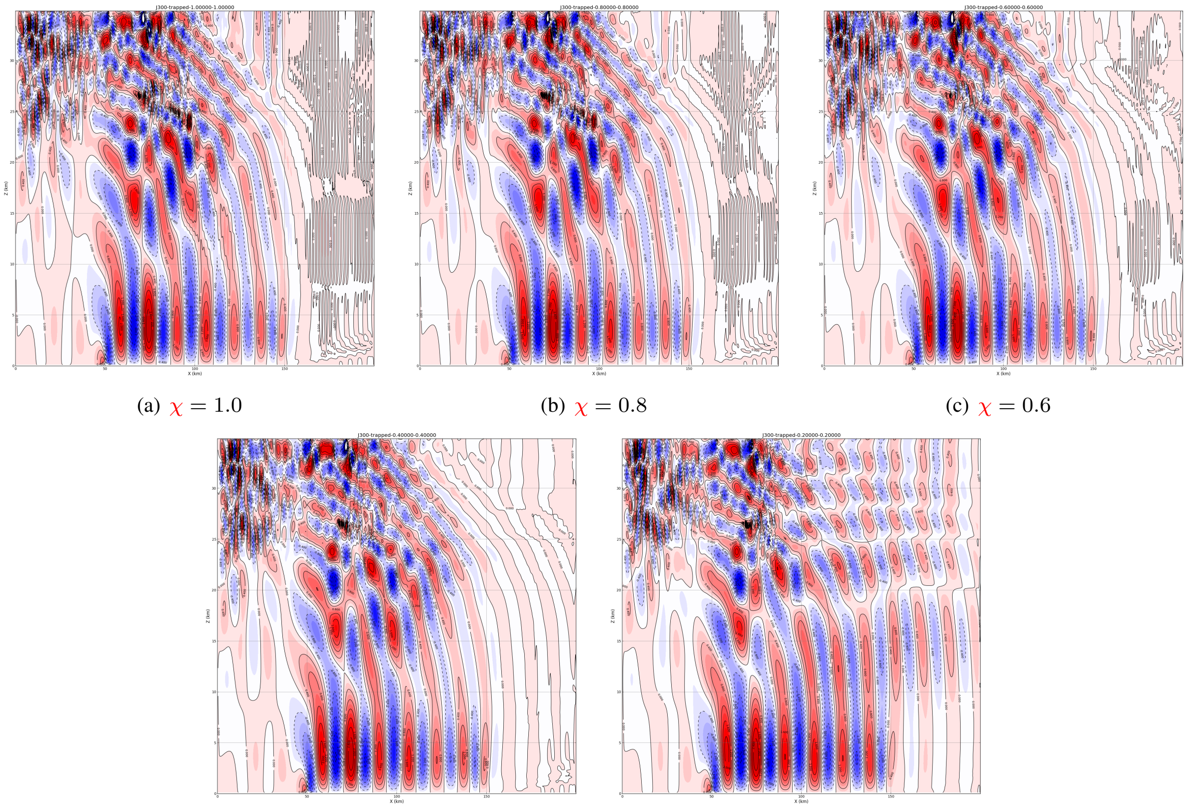
(wee confirmed partially results from Krusne Hory case, however due to limited time we decided that we will prepare rather parallel suite with NHHY and to test stability on other cases within that suite).

C. Very high resolution case study - Occitance

Finally we run the most difficult case - Occitance case study from 3rd of October 2015 00UTC. Domain contain Alps, see, flat land at resolution 375m. Here we presents results obtained after 12h of integration (this we consider sufficient to clarify

Table I: Parameters used in experiments that are changes with respect to reference experiment W001.

Variable	W001	W034	W042	W031	W027
LPC_FULL	T	F	F	F	F
NSITER	1	0	0	0	0
LPC_CHEAP	T	F	F	F	F
LNESC	T	F	F	F	F
LSETTLS	F	T	F	F	F
RDAMPPD	5	20	20	20	20
RDAMPVD	20	5	5	5	5
RDAMPVDS	15	10	10	10	10
LNHHY	F	T	T	T	T
RNHHY_VAL	1	1	0.95	0.7	0.5
RNHHY_VALLI	1	1.	1.05	1.0	0.5
RNHHY_GAMA_FAC	1	0	0	0	0
RNHHY_DELTA_FAC	1	2	2	2	2
NNHH_VERSION	-	1	1	1	1
RNHHY_TAU_NL	-	0	0	0	0



(a) $\chi = 1.0$ (b) $\chi = 0.8$ (c) $\chi = 0.6$

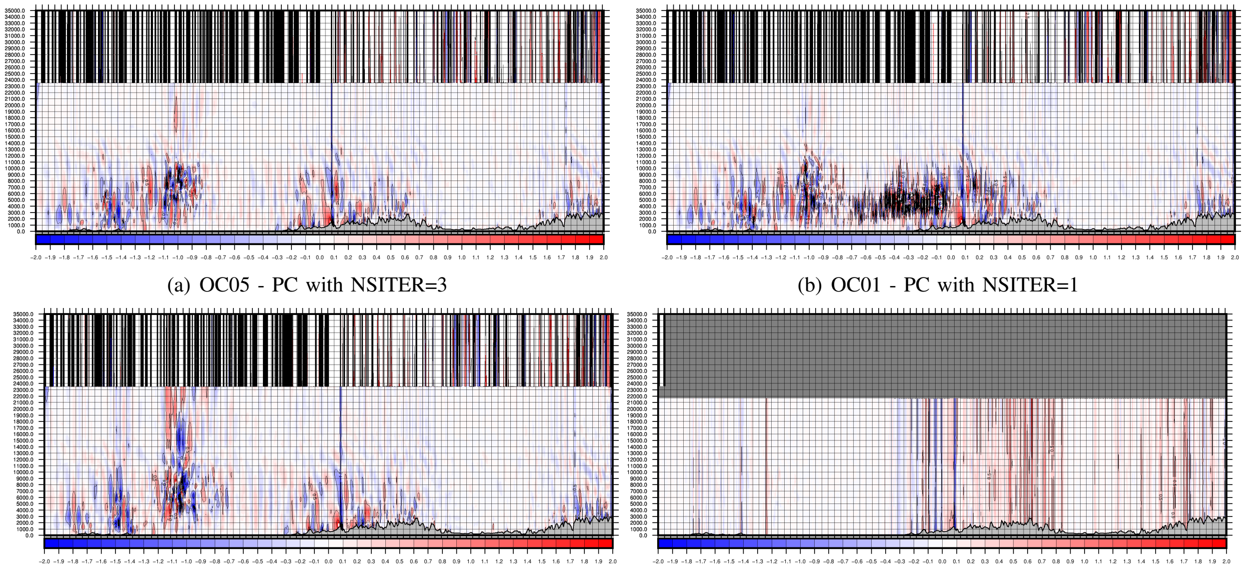
(d) $\chi = 0.4$ (e) $\chi = 0.2$

Figure 7: True vertical velocity w [m/s] in Keller trapped lee wave test case after 1000 time steps. Results are always obtain with the same value of χ parameter in both part of model and $\gamma = \chi$

time stepping scheme as stable).

We report results from 4 experiment

- OC05 - reference result - PC scheme with 3 iterations,
- OC01 - PC scheme with 1 iteration,



(a) OC05 - PC with NSITER=3

(b) OC01 - PC with NSITER=1

(c) OC00 - SI NHHY with $h = 0.3$

(d) OC03 - SI with HY model

Figure 8: True vertical velocity w [m/s] - diagonal cross section through Occitance domain

Table II: Parameters used in experiments that are changes with respect to reference experiment OC05.

Variable	OC05	OC01	OC00	OC03
LPC_FULL	T	T	F	F
NSITER	3	1	0	0
LPC_CHEAP	T	T	F	F
LNESC	T	T	F	F
LSETTLS	F	F	F	F
RDAMPPD	20	20	20	20
RDAMPVD	5	5	5	5
RDAMPVDS	10	10	10	10
LNHHY	F	F	T	T
RNHHY_VAL	1	1	0.3	0.0
RNHHY_VALLI	1	1	0.3	0.0
RNHHY_GAMA_FAC	1	1	0	0
RNHHY_DELTA_FAC	1	1	2	2
NNHH_VERSION	-	-	1	1
RNHHY_TAU_NL	-	-	0	0

- OC00 - SI SETTLS scheme with NHHY approximation in NL and LI model (both $h = 0.3$ resp. $\chi = 0.09$),
- OC03 - SI SETTLS scheme with HY model obtained with NHHY approximation in NL and LI model (both $h = 0$ resp. $\chi = 0$).

Setting of experiment namelist can be found in Table [II].

We presents diagonal cross section through domain on Figure (8). We see that currently operational PC scheme with 1 iterations (OC01) is unstable (not in the sense that model will blow, but spurious oscillation in middle troposphere are present). We therefore run PC scheme with 3 iterations. We obtained reference results free of oscillations.

The stabilization of 2TL SI SETTLS scheme for Occitance test case required introduction of consistent NHHY approximation into NL and LI part of model. Parameter $\chi = 0.09$ was needed to achieve stable SI scheme integration (OC00). We see that results are very close to reference except artifacts in stratospheric region.

To better understand our result we run the case with hydrostatic model (with used NHHY approximation with $\chi = 0$). We see that HY solution is very far from reference (vertical velocity was computed using omega equation).

We can conclude that it is possible to stabilize SI SETTLS scheme also for resolution 375m , but it requires very small values of NHHY parameters ($\chi = 0.09$). But as we shown at idealized tests during previous stay, the solution remains quite close to reference NH solution also in such cases. However, big problem is that in some part of region under such assumption strong vertical velocity may appear in stratosphere. We therefore propose following steps in our work in order to deal with that effect at resolution 375m:

1) test slope limited in order to smooth slopes that are too steep (there is less that 0.04% points with slope ¿ 1),
2) combine NHHY approximation with development of Francoise Voitus (he manage to stabilize spurious oscillation for OC01 case, by introduction of big W variable),
3) introduce vertically dependent NHHY parameters (we tried but so far it leads to very unstable time stepping), idea is needed how to proceed in design of such stable approach.

VI. IMPLEMENTATION

A. Implementation into linear part

In order to better understand SI computation here we conclude what is computed by SI routines.

$$
\begin{aligned}
SITNU_{PT}(PD) &= \kappa T^* \mathbf{S}^*(PD) \\
SITNU_{PSP}(PD) &= \mathbf{N}^*(PD) \\
SIGAM_{PD}(PT, PSP) &= R\mathbf{G}^*(PT) + RT^*(PSP) \\
SISEVE_{PV2}(PV1) &= \tfrac{1}{r}\mathbf{L}_v^* \gamma (PV1) \\
SIDD_{PDH}(PT, PSP, PRNH) &= R\mathbf{G}^*(PT) + RT^*(PSP) + RT^*(\beta - \mathbf{G}^*\epsilon)(PRNH) \\
SIDD_{PDV}(PRNH) &= \tfrac{g^2}{RT^*}\tfrac{1}{r}\mathbf{L}_v^* \gamma (PRNH) \\
SIPTP_{PT}(PDH, PDV) &= \tfrac{RT^*}{C_v}(PDH + PDV) \\
SIPTP_{PRNH}(PDH, PDV) &= \tfrac{C_p}{C_v}(PDH + PDV) - \mathbf{S}^*(PDH)
\end{aligned}
\tag{58}
$$

with $r = \frac{T_a^*}{T^*} = \frac{SITRA}{SITR}$ applied by default (can be off via optional argument).

We modified following computations

- SUNHEEBMAT - computation of $\mathbf{B}$ matrix (part of Helmholtz operator) and $\mathbf{H}_v^{*-1}$ operator
- ESPNHEESI - spectral SI computations, elimination, solution of Helmholtz and back-substitution
- LANHSI - linear part of grid point RHS

Implementation of summarized in the following tables.

Table III: Implementation SUNHEEBMAT

Variable	This report	with $\beta\delta = \chi$
ZSIB_HYD	$c^{*2}\frac{C_v}{C_p}(\kappa\mathbf{G}^*\mathbf{S}^* + \mathbf{N}^*)$	same as I
SIFACI	$\left(\mathbf{I} - \Delta t^2 c^{*2}\mathbf{L}_v^{**}\gamma\delta\right)^{-1} = \mathbf{H}_v^{*-1}$	$\left(\mathbf{I} - \Delta t^2 c^{*2}\mathbf{L}_v^{**}\gamma\delta\right)^{-1} = \mathbf{H}_v^{*-1}$
ZZ1	$c^{*2}\mathbf{I} - C_p\kappa T^*\mathbf{S}^* = c^{*2}\mathbf{S}_\kappa^*$	same as I
ZZ2	$\mathbf{H}_v^{*-1}c^{*2}\mathbf{S}_\kappa^*$	$\chi\mathbf{H}_v^{*-1}c^{*2}\mathbf{S}_\kappa^*$
ZZ3	$R\left(\mathbf{G}^*\chi\right)\mathbf{H}_v^{*-1}c^{*2}\mathbf{S}_\kappa^*$	$R\mathbf{G}^*\chi\mathbf{H}_v^{*-1}c^{*2}\mathbf{S}_\kappa^*$
ZSIB_ADD	$(\zeta\mathbf{I} - \frac{C_v}{C_pR}R\mathbf{G}^*\chi)\mathbf{H}_v^{*-1}c^{*2}\mathbf{S}_\kappa^*$	$(\mathbf{I} - \frac{C_v}{C_pR}R\mathbf{G}^*)\chi\mathbf{H}_v^{*-1}c^{*2}\mathbf{S}_\kappa^*$
	$= c^{*2}\left[\mathbf{G}_\kappa^*\chi + (\zeta - \chi)\mathbf{I}\right]\mathbf{H}_v^{*-1}\mathbf{S}_\kappa^*$	$= c^{*2}\mathbf{G}_\kappa^*\chi\mathbf{H}_v^{*-1}\mathbf{S}_\kappa^*$

Table IV: Implementation ESPNHEESI - solution of D

Variable	This report	with $\beta\delta = \chi$
ZSDIV	$R\mathbf{G}^*\tilde{T} + RT^*\tilde{q}_s + RT^*(\beta - \mathbf{G}^*\epsilon)\tilde{\hat{q}}$	$R\mathbf{G}^*\tilde{T} + RT^*\tilde{q}_s + RT^*(\beta - \mathbf{G}^*\epsilon)\tilde{\hat{q}}$
ZSVED	$\frac{g^2}{RT^*}\frac{1}{r}\mathbf{L}_v^*\gamma\tilde{\hat{q}}$	$\frac{g^2}{RT^*}\frac{1}{r}\mathbf{L}_v^*\gamma\tilde{\hat{q}}$
ZDH_DOT	$\tilde{D} - \Delta t\nabla^2 R\mathbf{G}^*\tilde{T} + RT^*\tilde{q}_s + RT^*(\beta - \mathbf{G}^*\epsilon)\tilde{\hat{q}} = D^\bullet$	$\tilde{D} - \Delta t\nabla^2 R\mathbf{G}^*\tilde{T} + RT^*\tilde{q}_s + RT^*(\beta - \mathbf{G}^*\epsilon)\tilde{\hat{q}} = D^\bullet$
ZVD_DOT	$\mathbf{I} - \Delta t\frac{g^2}{RT^*}\frac{1}{r}\mathbf{L}_v^*\gamma\tilde{\hat{q}} = d^\bullet$	$\mathbf{I} - \Delta t\frac{g^2}{RT^*}\frac{1}{r}\mathbf{L}_v^*\gamma\tilde{\hat{q}} = d^\bullet$
Z11	$\mathbf{H}_v^{*-1}d^\bullet$	$\chi\mathbf{H}_v^{*-1}d^\bullet$
Z12	$R\left(\mathbf{G}^*\chi\right)\mathbf{H}_v^{*-1}d^\bullet$	$R\mathbf{G}^*\chi\mathbf{H}_v^{*-1}d^\bullet$
ZSRHS	$D^\bullet - \Delta t^2\nabla^2\left[T^*R\left(\mathbf{G}^*\chi\right)\mathbf{H}_v^{*-1}d^\bullet - c^{*2}\zeta\mathbf{H}_v^{*-1}d^\bullet\right]$	$D^\bullet - \Delta t^2\nabla^2\left(T^*R\mathbf{G}^*\chi\mathbf{H}_v^{*-1}d^\bullet - c^{*2}\chi\mathbf{H}_v^{*-1}d^\bullet\right)$
	$= D^\bullet + \Delta t^2\nabla^2 c^{*2}\left[\mathbf{G}_\kappa^*\chi + (\zeta - \chi)\mathbf{I}\right]\mathbf{H}_v^{*-1}d^\bullet = D^{\bullet\bullet}$	$= D^\bullet + \Delta t^2\nabla^2 c^{*2}\mathbf{G}_\kappa^*\chi\mathbf{H}_v^{*-1}d^\bullet = D^{\bullet\bullet}$
ZZSPDIVG	$\left(\mathbf{I} - \Delta t^2\nabla^2\mathbf{B}\right)^{-1}D^{\bullet\bullet}$	same as I

Table V: Implementation ESPNHEESI - back-substitution of d

Variable	This report
Z21	$\delta\left(C_p\kappa T^*\mathbf{S}^* - c^{*2}\right)m^2D = -c^{*2}\delta\mathbf{S}_\kappa^*m^2D$
Z22	$-\frac{1}{r}\mathbf{L}_v^*\gamma c^{*2}\delta\mathbf{S}_\kappa^*m^2D$
Z23	$d^\bullet + \frac{\Delta t^2}{H^{*2}}\frac{1}{r}\mathbf{L}_v^*\gamma\delta c^{*2}\mathbf{S}_\kappa^*m^2D$
	$= d^\bullet + \Delta t^2c^{*2}\mathbf{L}_v^{**}\gamma\delta\mathbf{S}_\kappa^*m^2D$
ZZSPSVDG	$\mathbf{H}_v^{*-1}\left(d^\bullet + \Delta t^2c^{*2}\mathbf{L}_v^{**}\gamma\delta\mathbf{S}_\kappa^*m^2D\right)$

Table VI: Implementation ESPNHEESI - back-substitution of $\hat{q}$

Variable	This report
ZZSPSPDG	$\tilde{\hat{q}} - \Delta t\delta\left[\frac{C_p}{C_v}\left(m^2D + d\right) - \frac{C_p}{RT^*}\kappa T^*\mathbf{S}^*m^2D\right]$
	$\tilde{\hat{q}} - \Delta t\delta\left[\frac{C_p}{C_v}\left(m^2D + d\right) - \mathbf{S}^*m^2D\right]$

Table VII: Implementation ESPNHEESI - back-substitution of T

Variable	This report
ZZSPTG	$\tilde{T} - \Delta t\alpha\frac{RT^*}{C_v}\left(m^2D + d\right) - \Delta t\left(1 - \alpha\right)\kappa T^*\mathbf{S}^*m^2D$

Table VIII: Computation of geopotentiel

Variable	Routine	Term
TO EDIT	CPG_GP_NHEE	$\tilde{T} - \Delta t\alpha\frac{RT^*}{C_v}\left(m^2D + d\right) - \Delta t\left(1 - \alpha\right)\kappa T^*\mathbf{S}^*m^2D$

B. *Implementation into nonlinear part*

VII. Conclusion

We introduced new broad set of parameters into dynamical core of ALADIN - we call them NHHY parameters. We did that consistently in NL part of model and also LI part of model. However, we allowed set of parameters to differ in NL and LI model.

At resolution 2.3km we managed stabilize SI SETTLS scheme with introduction of NHHY parameters into linear part of model only. However, we found that for resolution 375m this is not sufficient. Therefore the question that have to be answered is, for which resolution our NHHY approximation can stabilize NH model with unapproximated nonlinear part.

We further found that at resolution 375m the stabilization of SI SETTLS scheme is possible, only when we use NHHY parameters close to 0. But we showed that despite such approximation the character of solution remains very close to reference results. However, due to lack of time we did not quantify errors. This have to be done in order to presents more rigorous and precise conclusion.

During last 2 stays (3 months of work) we formulated scheme, we derive characteristic equation (thanks to P. Smolikova), we prepare analysis of stability into full model code, we implemented, debug and validated our idea, we performed idealized tests and real 3d tests.

VIII. Appendix A

Taking into account that NH additional part shall be $c^{*2}\mathbf{G}^*_\kappa\mathbf{H}^{*-1}_v\mathbf{S}^*_\kappa$ we were surprised by the length of original code in CY47 in routine SUNHEEBMAT.

We analyzed the computation as implemented in reference version

$$\begin{aligned}
ZZ1 &= c^{*2}\mathbf{I} - C_p\kappa T^*\mathbf{S}^* = c^{*2}\mathbf{S}^*_\kappa \\
ZZ21 &= c^{*2}\tfrac{1}{r}\mathbf{L}^*_v\mathbf{S}^*_\kappa \\
ZZ22 &= c^{*2}\tfrac{1}{r}\mathbf{H}^{*-1}_v\mathbf{L}^*_v\mathbf{S}^*_\kappa \\
ZZ2 &= c^{*2}\left(\Delta t^2 c^{*2}\mathbf{H}^{*-1}_v\mathbf{L}^{**}_v + \mathbf{I}\right)\mathbf{S}^*_\kappa \\
ZZ3 &= Rc^{*2}\mathbf{G}^*\left(\Delta t^2 c^{*2}\mathbf{H}^{*-1}_v\mathbf{L}^{**}_v + \mathbf{I}\right)\mathbf{S}^*_\kappa \\
ZSIB_ADD &= \left(\mathbf{I} - \tfrac{C_v}{C_p}\mathbf{G}^*\right)c^{*2}\left(\Delta t^2 c^{*2}\mathbf{H}^{*-1}_v\mathbf{L}^{**}_v + \mathbf{I}\right)\mathbf{S}^*_\kappa \\
&= c^{*2}\mathbf{G}^*_\kappa\left(\Delta t^2 c^{*2}\mathbf{H}^{*-1}_v\mathbf{L}^{**}_v + \mathbf{I}\right)\mathbf{S}^*_\kappa \\
&= c^{*2}\mathbf{G}^*_\kappa\left[\mathbf{H}^{*-1}_v\left(\mathbf{I} - \mathbf{H}^*_v\right) + \mathbf{I}\right]\mathbf{S}^*_\kappa \\
&= c^{*2}\mathbf{G}^*_\kappa\mathbf{H}^{*-1}_v\mathbf{S}^*_\kappa
\end{aligned} \tag{59}$$

We considered this implementation to be very complicated for understanding. Therefore we have implemented simpler version without any need to compute vertical laplacian operator $\mathbf{L}^*_v$. The new version is in Table III.

We perform validation. The new simpler implementation slightly differ at the last digit of spectral norms.