NH system as departure from HY system. Unification of HY and NH code.

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December 6, 2019

Abstract

Currently HY and NH dynamical cores forms two different worlds in the dynamical core. Recently Voitus showed that the spectral computations of NH model can be treated as departure from HY model. The same approach can be adopted for all grid point computations. We can introduce new quantity ϵ that will control non-hydrostaticity of dynamical core ($\epsilon = 1$ NH core, $\epsilon = 0$ HY core).
ϵ can be vertically dependent. It allow us to supress non-hydrostatism close to model top where vertical resolution is too coarse to properly sample NH processes. NH close to model top causes problems with stability for years.
In this memo I formulate model with ϵ and I propose reformulation of dynamics.

1 Methodology

Our aim was to introduce control paramatere $\epsilon = \epsilon(\eta)$ in such a way that we can control hydrostaticity in upper part of model domain. The way how to define such parameter is not unique. We require that for $\epsilon = 0$ system will be hydrostatic in whole domain and for $\epsilon = 1$ system will provide non-hydrostatic solution. The mixed state in between is not physically defined, it can be any system that satisfies limit cases.

Our first attempts were based on the fact that we have hydrostatic pressure π available in our system. In the case of HY model the true pressure $p = \pi$. In NH model, the true pressure p is defined via diagnostic relation $p = \pi e^{\hat{q}}$ and the time change of true pressure $\frac{dp}{dt} = \dot{p}$ is given

$$\frac{\dot{p}}{p} = \frac{\dot{\pi}}{\pi} + \dot{\hat{q}} = \frac{\omega}{\pi} + \dot{\hat{q}}. \tag{1}$$

Natural idea how to control hydrostaticity is to limit values of $\hat{q}$ in one of the following ways

1. $p = \pi e^{\epsilon \hat{q}}$
2. $p = \pi + \epsilon \pi \left(e^{\hat{q}} - 1\right)$.

However, it can be easily showed that such definitions leads to singularity when $\epsilon = 0$. Prognostic euation for $\hat{q}$ than becomes diagnostic one for D_3 term that appears in equation for T.

Therefore we look for different way to introduce parameter ϵ. We finally found the way that we present in this report. It would be difficult to give explanation with physically relevant reasoning.

2 Nonlinear model

We use prognostic quantities of model ALADIN $\vec{v}, T, q_s = ln(\pi_s), w, \hat{q} = ln(\frac{p}{\pi})$.

We start with the definition of mass coordinate π (lets call it hydrostatic pressure in the presence of gravity)

$$\frac{\partial \phi}{\partial \pi} = -\frac{RT}{p}. \tag{2}$$

We rewrite definition of π in η and we introduce ϵ parameter that controls NH part of relation.

$$\frac{\partial \phi}{\partial \eta} = -\frac{mRT}{p} = -\frac{mRT}{\pi} - \epsilon \left(\frac{\pi}{p} - 1\right) \frac{mRT}{\pi}. \tag{3}$$

with $\frac{p}{\pi} = e^{\hat{q}}$ and $\frac{\pi}{p} - 1 = \frac{1-e^{\hat{q}}}{e^{\hat{q}}}$.

Evolution of T is defined in the following way

$$\frac{dT}{dt} = (1 - \epsilon^2)\frac{\kappa T \omega}{\pi} - \epsilon^2 \frac{RT}{c_v} D_3. \tag{4}$$

The reason why the square ϵ^2 appears in T equation will become clear during elimination of SI linear system. Such definition allows to keep maximum consistency with already existing system.

Evolution of $\hat{q}$ is defined as

$$\frac{d\hat{q}}{dt} = -\epsilon \left(\frac{c_p}{c_v} D_3 + \frac{\omega}{\pi}\right). \tag{5}$$

Horizontal momentum equation in non-rotating frame (coriolis terms not included) in η coordinate gives

$$\begin{aligned} \frac{d\vec{v}}{dt} &= -RT\frac{\vec{\nabla}\pi}{\pi} - \vec{\nabla}\phi \\ &+\epsilon\left(-RT\vec{\nabla}\hat{q} - \left(\frac{1}{m}\frac{\partial p}{\partial \eta} - 1\right)\vec{\nabla}\phi\right) \end{aligned} \tag{6}$$

We would like to emphasize at this moment that term $\left(\frac{1}{m}\frac{\partial p}{\partial \eta} - 1\right)\vec{\nabla}\phi$ is highly nonlinear ans it has no equivalent terms in current SI operator. There is hidden NH term inside $\vec{\nabla}\phi$ as computation of geopotential itself contains ϵ parameter (see [11]).

Vertical momentum equation gives

$$\frac{dgw}{dt} = \epsilon g^2\left(\frac{1}{m}\frac{\partial p}{\partial \eta} - 1\right) \tag{7}$$

in both equations above $\frac{1}{m}\frac{\partial p}{\partial \eta} - 1 = e^{\hat{q}} - 1 + \frac{p}{m}\frac{\partial \hat{q}}{\partial \eta}$.

We use prognostic quantity d in spectral space due to stability reasons (the same reasons why we use horizontal divergence and vorticity instead wind components). If we introduce ϵ in the previous way into w equation than nonlinear evolution of d is given by equation

$$\frac{dd}{dt} = -g^2\frac{p}{mRT}\frac{\partial}{\partial \eta}\left[\epsilon\frac{1}{m}\frac{\partial(p-\pi)}{\partial \eta}\right] + ... \tag{8}$$

We write only leading vertical laplacian term, which has conuterpart in SI system. For complete equation see documentation of NH equation [2.37].

The question to be exploited is, if so called X-term shall also involve ϵ or not.Remanins to be explored.

Following set of equations summarizes our nonlinear system.

$$\begin{aligned} \frac{\partial q_s}{\partial t} &= -\frac{1}{\pi_s}\int_0^1 \vec{\nabla}(m\vec{v})d\eta \\ \frac{dT}{dt} &= \frac{\kappa T\omega}{\pi} - \epsilon^2\left(\frac{\kappa T\omega}{\pi} + \frac{RT}{c_v}D_3\right) \\ \frac{d\vec{v}}{dt} &= -RT\frac{\vec{\nabla}\pi}{\pi} - \vec{\nabla}\phi - \epsilon\left[RT\vec{\nabla}\hat{q} + \left(\frac{1}{m}\frac{\partial p}{\partial \eta} - 1\right)\vec{\nabla}\phi\right] \\ \frac{dgw}{dt} &= \epsilon g^2\left(\frac{1}{m}\frac{\partial p}{\partial \eta} - 1\right) \\ \frac{d\hat{q}}{dt} &= -\epsilon\left(\frac{c_p}{c_v}D_3 + \frac{\omega}{\pi}\right) \end{aligned} \tag{9}$$

Black terms are HY system and red terms represents NH increment to original HY system. When $\epsilon = 0$ we obtain hydrostatic set of equations.

The system is closed with diagnostic relations

$$
\begin{aligned}
\omega &= \vec{v}\vec{\nabla}\pi - \int_0^{\eta} \vec{\nabla}\left(m\vec{v}\right) d\eta & (10)\\
\phi &= \phi_s - \int_{\eta}^{1} \frac{mRT}{\pi} d\eta^{'} \\
& \quad - \int_{\eta}^{1} \epsilon \left(\frac{\pi}{p} - 1\right) \frac{mRT}{\pi} d\eta^{'} & (11)\\
D_3 &= D + d & (12)\\
d &= -\frac{p}{RT}\frac{1}{m}\frac{\partial gw}{\partial \eta} + \epsilon_3 \frac{p}{RT}\vec{\nabla}\phi\frac{1}{m}\frac{\partial \vec{v}}{\partial \eta} & (13)
\end{aligned}
$$

The paramater ϵ_3 is applied X-term and it has no counterpart in SI linear system.

3 Linear mode and elimination of variables I.

When [9] is linearized around bi-isothermal, resting, horizontally homogennous state defined by $T^*, {T_a}^* {\pi_s}^*$ we obtain following set of linearized equations

$$
\begin{aligned}
\frac{\partial q_s}{\partial t} &= -\mathbf{N}^* D \\
\frac{\partial T}{\partial t} &= -\kappa T^* \mathbf{S}^* D + \epsilon^2 \left[\kappa T^* \mathbf{S}^* D - \frac{RT^*}{C_v}\left(D + d\right)\right] \\
\frac{\partial D}{\partial t} &= -R\mathbf{G}^* \triangle T - RT^* \triangle q_s - \triangle \phi_s \\
& \quad + RT^* (\mathbf{G}^* - 1)\epsilon \triangle \hat{q} \\
\frac{\partial d}{\partial t} &= -\frac{g^2}{R{T_a}^*}\mathbf{L}_v^* \hat{q} \\
\frac{\partial \hat{q}}{\partial t} &= -\epsilon \frac{C_p}{C_v}(D + d) + \epsilon \mathbf{S}^* D.
\end{aligned}
\qquad (14)
$$

The operators $\mathbf{S}^*, \mathbf{G}^*, \mathbf{N}^*, \mathbf{L}_v^*$ are defined as

$$
\begin{aligned}
\mathbf{G}^* X &= \int_{\eta}^{1} \frac{m^*}{\pi^*} X d\eta \\
\mathbf{S}^* X &= \frac{1}{\pi^*} \int_0^{\eta} m^* X d\eta \\
\mathbf{N}^* X &= \frac{1}{{\pi_s}^*} \int_0^{1} m^* X d\eta \\
\mathbf{L}_v^* X &= \frac{\pi^*}{m^*}\frac{\partial}{\partial \eta}\left[\epsilon\left(\frac{\pi^*}{m^*}\frac{\partial}{\partial \eta} + 1\right)\right] X
\end{aligned}
$$

Here we perform detailed elimination assuming that parameter $\epsilon = \epsilon(\eta)$ can vary in vertical.

We apply time derivatie on D equation and we take into account that ϕ_s, ϵ, T^*, R are time indepenedent. I work with the assumption that ϵ is inside geopotential integral in NL system.

$$\frac{\partial^2 D}{\partial t^2} = -R\mathbf{G}^*\triangle\frac{\partial T}{\partial t} - RT^*\triangle\frac{\partial q_s}{\partial t} + RT^*(\mathbf{G}^*-1)\epsilon\triangle\frac{\partial \hat{q}}{\partial t} \tag{15}$$

and we subsitute from [38]. We obtain

$$\begin{aligned}\frac{\partial^2 D}{\partial t^2} = & -R\mathbf{G}^*\triangle\left\{-\kappa T^*\mathbf{S}^*D + \epsilon^2\left[\kappa T^*\mathbf{S}^*D - \frac{RT^*}{C_v}(D+d)\right]\right\} \\ & -RT^*\triangle\{-\mathbf{N}^*D\} \\ & +RT^*(\mathbf{G}^*-1)\epsilon\triangle\left\{-\epsilon\frac{C_p}{C_v}(D+d)+\epsilon\mathbf{S}^*D\right\}\end{aligned} \tag{16}$$

after manipulation and taking into account $gH^* = RT^*$ we can write

$$\begin{aligned}\frac{\partial^2 D}{\partial t^2} = & -gH^*\mathbf{G}^*\left\{-\kappa\mathbf{S}^*\triangle D + \epsilon^2\left[\kappa\mathbf{S}^*\triangle D - \frac{R}{C_v}(\triangle D+\triangle d)\right]\right\} \\ & -gH^*\{-\mathbf{N}^*\triangle D\} \\ & +gH^*(\mathbf{G}^*-1)\epsilon\left\{-\epsilon\frac{C_p}{C_v}(\triangle D+\triangle d)+\epsilon\mathbf{S}^*\triangle D\right\}\end{aligned} \tag{17}$$

After additional manipulation we obtain

$$\begin{aligned}\frac{\partial^2 D}{\partial t^2} = & \; gH^*\kappa\mathbf{G}^*\mathbf{S}^*\triangle D + gH^*\mathbf{N}^*\triangle D \\ & -gH^*\kappa\mathbf{G}^*\epsilon^2\mathbf{S}^*\triangle D \\ & +gH^*\frac{R}{C_v}\mathbf{G}^*\epsilon^2\triangle D + gH^*\frac{R}{C_v}\mathbf{G}^*\epsilon^2\triangle d \\ & -gH^*\frac{C_p}{C_v}\mathbf{G}^*\epsilon^2\triangle D + gH^*\frac{C_p}{C_v}\epsilon^2\triangle D \\ & -gH^*\frac{C_p}{C_v}\mathbf{G}^*\epsilon^2\triangle d + gH^*\frac{C_p}{C_v}\epsilon^2\triangle d \\ & +gH^*\mathbf{G}^*\epsilon^2\mathbf{S}^*\triangle D - gH^*\epsilon^2\mathbf{S}^*\triangle D.\end{aligned} \tag{18}$$

First two terms are HY terms. Additional terms are NH departure and taking into account $c^{*2} = \frac{C_p}{C_v}RT^*$ and $\frac{C_p}{C_v} - \frac{R}{C_v} = 1$ and $\frac{R}{C_p} = 1 - \frac{C_v}{C_p}$

$$\begin{aligned}\frac{\partial^2 D}{\partial t^2} = & \; gH^*\kappa\mathbf{G}^*\mathbf{S}^*\triangle D + gH^*\mathbf{N}^*\triangle D \\ & +gH^*\left(\frac{C_v}{C_p}\mathbf{G}^*\epsilon^2\mathbf{S}^* - \mathbf{G}^*\epsilon^2 - \epsilon^2\mathbf{S}^*\right)\triangle D \\ & +gH^*\left(\frac{R}{C_p}\mathbf{G}^*\epsilon^2\mathbf{S}^* - \frac{R}{C_p}\mathbf{G}^*\epsilon^2\mathbf{S}^* + \epsilon_s^2\mathbf{N}^* - \epsilon_s^2\mathbf{N}^*\right)\triangle D \\ & +c^{*2}\epsilon^2\triangle D \\ & +(-gH^*\mathbf{G}^* + c^{*2})\epsilon^2\triangle d\end{aligned} \tag{19}$$

We showed already in 2006 (report of Vivoda) that

$$\mathbf{A}_1^* := \mathbf{G}^*\mathbf{S}^*X - \mathbf{G}^*X - \mathbf{S}^*X + \mathbf{N}^*X = 0$$

has analogous relation when vertically dependent function τ is considered

$$\mathbf{A}_1^* := \mathbf{G}^*\tau(\eta)\mathbf{S}^*X - \mathbf{G}^*\tau(\eta)X - \tau(\eta)\mathbf{S}^*X + \tau_s\mathbf{N}^*X = \mathbf{G}^*\frac{\pi^*}{m^*}\frac{\partial\tau(\eta)}{\partial\eta}\mathbf{S}^*X. \tag{20}$$

In order to employ previous relation we added red terms in [44]. After manipulations and using [45] with $\tau = \epsilon^2$ we obtain

$$\begin{aligned}\frac{\partial^2 D}{\partial t^2} &= gH^*\left(\kappa\mathbf{G}^*\mathbf{S}^* + \mathbf{N}^*\right)\triangle D \\ &\quad + \left[gH^*\ \left(\mathbf{A}_1^* - \kappa\mathbf{G}^*\epsilon^2\mathbf{S}^* - \epsilon_s^2\mathbf{N}^*\right) + c^{*2}\epsilon^2\right]\triangle D \\ &\quad + (-gH^*\mathbf{G}^* + c^{*2})\epsilon^2\triangle d\end{aligned} \tag{21}$$

The same procedure we repeat with equation for vertical divergence

$$\frac{\partial^2 d}{\partial t^2} = \frac{1}{rH^{*2}}\mathbf{L}_v^*\left(-gH^*\epsilon\mathbf{S}^* + c^{*2}\epsilon\right)D + c^{*2}\frac{1}{rH^{*2}}\mathbf{L}_v^*\epsilon d \tag{22}$$

with $r = \frac{T_a{}^*}{T^*}$, $c^{*2} = \frac{c_p}{c_v}RT^*$ and $H^* = \frac{RT^*}{g}$.

We can noticed that for $\epsilon = 0$ we obtain from [46] HY Helmholtz equation

$$\frac{\partial^2 D}{\partial t^2} = gH^*\left(\kappa\mathbf{G}^*\mathbf{S}^* + \mathbf{N}^*\right)\triangle D \tag{23}$$

and for the $\epsilon = 1$ NH equation as reported by Benard (NH documentation).

$$\frac{\partial^2 D}{\partial t^2} = \left(-gH^*\mathbf{A}_1^* + c^{*2}\right)\triangle D + \left(-gH^*\mathbf{G}^* + c^{*2}\right)\triangle d \tag{24}$$

When system is discretized in time and in vertical direction, we can follow steps reported in Voitus report. Our implicit system to be solved is

$$\begin{aligned} q_s - \delta t\left(-\mathbf{N}^*m^2D\right) &= q_{sRHS} \\ T - \delta t\left\{-\kappa T^*\mathbf{S}^*m^2D + \epsilon^2\left[\kappa T^*\mathbf{S}^*m^2D - \frac{RT^*}{C_v}\left(m^2D + d\right)\right]\right\} &= T_{RHS} \\ D - \delta t\left[-R\mathbf{G}^*\triangle T - RT^*\triangle q_s - \triangle\phi_s + RT^*(\mathbf{G}^* - 1)\epsilon\triangle\hat{q}\right] &= D_{RHS} \\ d - \delta t\left(-\frac{g}{rH^*}\mathbf{L}_v^*\hat{q}\right) &= d_{RHS} \\ \hat{q} - \delta t\left[-\epsilon\frac{C_p}{C_v}(m^2D + d) + \epsilon\mathbf{S}^*m^2D\right] &= \hat{q}_{RHS}. \end{aligned} \tag{25}$$

We introduced map factor m to keep consistency with Voitus.

We first adopt operators introduced in his report

$$\begin{aligned} \mathbf{G}^*_\kappa X &= \mathbf{I} - \tfrac{c_v}{c_p}\mathbf{G}^* X \\ \mathbf{S}^*_\kappa X &= \mathbf{I} - \tfrac{c_v}{c_p}\mathbf{S}^* X \\ \mathbf{L}^{**}_v X &= \tfrac{1}{rH^{*2}}\mathbf{L}^*_v X \end{aligned}$$

The opeator $\mathbf{L}^{**}_v$ is vertically discretized version of continuous operator

$$\frac{1}{rH^{*2}}\frac{\pi^*}{m^*}\frac{\partial}{\partial\eta}\left[\epsilon\left(\frac{\pi^*}{m^*}\frac{\partial}{\partial\eta}+1\right)\right]$$

We can rewrite our system [50] using previous operators as

$$\begin{aligned} \frac{\partial^2 D}{\partial t^2} &= \left[c^{*2}\mathbf{B}_{HY} + c^{*2}(\epsilon^2\mathbf{I} + \mathbf{C}_1 - \mathbf{B}_b)\right]\triangle D \\ &\quad + c^{*2}\mathbf{G}^*_\kappa\epsilon^2\triangle d \\ \frac{\partial^2 d}{\partial t^2} &= c^{*2}\mathbf{L}^{**}_v\epsilon\mathbf{S}^*_\kappa D + c^{*2}\mathbf{L}^{**}_v\epsilon d \end{aligned} \tag{26}$$

With $\mathbf{C}_1 = -\frac{c_v}{c_p}\mathbf{A}^*_1$ and $\mathbf{B}_{HY} = \frac{c_v}{c_p}\left(\kappa\mathbf{G}^*\mathbf{S}^* + \mathbf{N}^*\right)$ and $\mathbf{B}_b = \frac{c_v}{c_p}\left(\kappa\mathbf{G}^*\epsilon^2\mathbf{S}^* + \epsilon_s^2\mathbf{N}^*\right)$. We have to realize that $\mathbf{C}_1$ in our case is discretized version of

$$\mathbf{C}_1 = \frac{C_v}{C_p}\left[\mathbf{G}^*\epsilon^2\mathbf{S}^*X - \mathbf{G}^*\epsilon^2 X - \epsilon^2\mathbf{S}^*X + \epsilon_s^2\mathbf{N}^*X\right].$$

When we assume also that the system is discretized in time we obtain equivalent of equations (9) and (10) from report of Voitus,

$$\begin{aligned} \left[\mathbf{I} - \delta t^2 c^{*2}\mathbf{B}_{HY} - \delta t^2 c^{*2}(\epsilon^2\mathbf{I} + \mathbf{C}_1 - \mathbf{B}_b)\right] m^2\triangle D & \\ - \delta t^2 c^{*2}\mathbf{G}^*_\kappa\epsilon^2\triangle d &= D^\bullet \\ \left[\mathbf{I} - \delta t^2 c^{*2}\mathbf{L}^{**}_v\epsilon\right] d - \delta t^2 c^{*2}\mathbf{L}^{**}_v\epsilon\mathbf{S}^*_\kappa D &= d^\bullet \end{aligned} \tag{27}$$

with (11) and (12) of Voitus being modified as

$$D^\bullet = D_{RHS} - \delta t R T^*\triangle\left[(\mathbf{I} - \mathbf{G}^*)\epsilon\hat{q}_{RHS} + \mathbf{G}^*\frac{T_{RHS}}{T^*} + q_{sRHS}\right] \tag{28}$$

and

$$d^\bullet = d_{RHS} - \delta t R T^*\mathbf{L}^{**}_v\hat{q}_{RHS}. \tag{29}$$

When we define vertical new vertical operator as

$$\mathbf{H}_v^* = \mathbf{I} - \delta t^2 c^{*2} \mathbf{L}_v^{**} \epsilon$$

and we modify (14) from Voitus as

$$d = \mathbf{H}_v^{*-1} \left[d^{\bullet} + \delta t^2 c^{*2} \mathbf{L}_v^{**} \epsilon \mathbf{S}_\kappa^* D \right]. \tag{30}$$

then we obtain

$$\begin{aligned}
\left[\mathbf{I} - \delta t^2 c^{*2} \mathbf{B}_{HY} - \delta t^2 c^{*2} \triangle (\epsilon^2 \mathbf{I} + \mathbf{C}_1 - \mathbf{B}_b) \right] m^2 D &= \\
D^{\bullet} + \delta t^2 c^{*2} \triangle \mathbf{G}_\kappa^* \epsilon^2 \mathbf{H}_v^{*-1} \left[d^{\bullet} + \delta t^2 c^{*2} \mathbf{L}_v^{**} \epsilon \mathbf{S}_\kappa^* D \right] &= \\
D^{\bullet\bullet} + \delta t^4 c^{*4} \triangle \mathbf{G}_\kappa^* \epsilon^2 \mathbf{H}_v^{*-1} \mathbf{L}_v^{**} \epsilon \mathbf{S}_\kappa^* m^2 D &
\end{aligned} \tag{31}$$

with

$$D^{\bullet\bullet} = D^{\bullet} + \delta t^2 c^{*2} \triangle \mathbf{G}_\kappa^* \epsilon^2 \mathbf{H}_v^{*-1} d^{\bullet}. \tag{32}$$

Resulting Helmholtz equation then yields

$$\left[1 - \delta t^2 \mathbf{B} \triangle \right] D = D^{\bullet\bullet} \tag{33}$$

with

$$\mathbf{B} = c^{*2} \left[\mathbf{B}_{HY} + \left(\epsilon^2 \mathbf{I} + \mathbf{C}_1 - \mathbf{B}_b + \delta t^2 c^{*2} \mathbf{G}_\kappa^* \epsilon^2 \mathbf{H}_v^{*-1} \mathbf{L}_v^{**} \epsilon \mathbf{S}_\kappa^* \right) \right]. \tag{34}$$

It can be shown that NH part can be rewritten as

$$\begin{aligned}
\epsilon^2 \mathbf{I} + \mathbf{C}_1 - \mathbf{B}_b + \delta t^2 c^{*2} \mathbf{G}_\kappa^* \epsilon^2 \mathbf{H}_v^{*-1} \mathbf{L}_v^{**} \epsilon \mathbf{S}_\kappa^* &= \\
\epsilon^2 \mathbf{I} + \mathbf{C}_1 - \mathbf{B}_b + \mathbf{G}_\kappa^* \epsilon^2 \mathbf{H}_v^{*-1} (1 - \mathbf{H}_v^*) \mathbf{S}_\kappa^* &= \\
\epsilon^2 \mathbf{I} + \mathbf{C}_1 - \mathbf{B}_b + \mathbf{G}_\kappa^* \epsilon^2 \mathbf{H}_v^{*-1} \mathbf{S}_\kappa^* - \mathbf{G}_\kappa^* \epsilon^2 \mathbf{S}_\kappa^* &= \\
\epsilon^2 \mathbf{I} + \tfrac{C_v}{C_p} \mathbf{G}^* \epsilon^2 \mathbf{S}^* - \tfrac{C_v}{C_p} \mathbf{G}^* \epsilon^2 - \tfrac{C_v}{C_p} \epsilon^2 \mathbf{S}^* + \tfrac{C_v}{C_p} \epsilon_s^2 \mathbf{N}^* & \\
- \tfrac{c_v}{c_p} \kappa \mathbf{G}^* \epsilon^2 \mathbf{S}^* - \tfrac{c_v}{c_p} \epsilon_s^2 \mathbf{N}^* & \\
- \epsilon^2 \mathbf{I} + \tfrac{C_v}{C_p} \epsilon^2 \mathbf{S}^* + \tfrac{C_v}{C_p} \mathbf{G}^* \epsilon^2 - \left(\tfrac{C_v}{C_p} \right)^2 \mathbf{G}^* \epsilon^2 \mathbf{S}^* & \\
+ \mathbf{G}_\kappa^* \epsilon^2 \mathbf{H}_v^{*-1} \mathbf{S}_\kappa^* &= \\
\tfrac{C_v}{C_p} \left(1 - \kappa - \tfrac{C_v}{C_p} \right) \mathbf{G}^* \epsilon^2 \mathbf{S}^* + \mathbf{G}_\kappa^* \epsilon^2 \mathbf{H}_v^{*-1} \mathbf{S}_\kappa^* &= \mathbf{G}_\kappa^* \epsilon^2 \mathbf{H}_v^{*-1} \mathbf{S}_\kappa^*
\end{aligned} \tag{35}$$

when using $\delta t^2 c^{*2} \mathbf{L}_v^{**} \epsilon = \mathbf{I} - \mathbf{H}_v^*$ and $1 - \kappa - \frac{C_v}{C_p} = 0$.

After re-arrangment we can show finally that

$$\mathbf{B} = c^{*2}\left(\mathbf{B}_{HY} + \mathbf{B}_{NH}\right). \tag{36}$$

with

$$B_{NH} = \mathbf{G}^*_{\kappa}\epsilon^2\mathbf{H}^{*-1}_{v}\mathbf{S}^*_{\kappa}. \tag{37}$$

4 Linear mode and elimination of variables II.

This is version 2 of the system, implemented in a such a way that we enforce $p = \pi e^{\epsilon\hat{q}}$. However, we did not change prognostic equation for $\hat{q}$. In the nonlinear model we do nothing else than we in routine GNHPRE define pressure using previous diagnostic relation. This leads to following SI linear model

$$\begin{aligned}
\frac{\partial q_s}{\partial t} &= -\mathbf{N}^* D \\
\frac{\partial T}{\partial t} &= -\kappa T^*\mathbf{S}^* D + \epsilon\left[\kappa T^*\mathbf{S}^* D - \frac{RT^*}{C_v}(D+d)\right] \\
\frac{\partial D}{\partial t} &= -R\mathbf{G}^*\triangle T - RT^*\triangle q_s - \triangle\phi_s \\
&\quad +RT^*(\mathbf{G}^*-1)\epsilon\triangle\hat{q} \\
\frac{\partial d}{\partial t} &= -\frac{g^2}{RT_a{}^*}\mathbf{V}^*_v\hat{q} \\
\frac{\partial \hat{q}}{\partial t} &= -\frac{C_p}{C_v}(D+d) + \mathbf{S}^* D.
\end{aligned} \tag{38}$$

The operators $\mathbf{L}^*_v$ is replaced by new operator $\mathbf{V}^*_v$. It is defined as

$$\mathbf{V}^*_v X = \frac{\pi^*}{m^*}\frac{\partial}{\partial\eta}\left[\epsilon\left(\frac{\pi^*}{m^*}\frac{\partial}{\partial\eta}+1\right) + \frac{\pi^*}{m^*}\left(\frac{\partial\epsilon}{\partial\eta}\right)\right] X \tag{39}$$

We follows the same steps as in section before

$$\frac{\partial^2 D}{\partial t^2} = -R\mathbf{G}^*\triangle\frac{\partial T}{\partial t} - RT^*\triangle\frac{\partial q_s}{\partial t} + RT^*(\mathbf{G}^*-1)\epsilon\triangle\frac{\partial\hat{q}}{\partial t} \tag{40}$$

and we substitute from [38]. We obtain

$$\begin{aligned}
\frac{\partial^2 D}{\partial t^2} &= -R\mathbf{G}^*\triangle\left\{-\kappa T^*\mathbf{S}^* D + \epsilon\left[\kappa T^*\mathbf{S}^* D - \frac{RT^*}{C_v}(D+d)\right]\right\} \\
&\quad -RT^*\triangle\left\{-\mathbf{N}^* D\right\} \\
&\quad +RT^*(\mathbf{G}^*-1)\epsilon\triangle\left\{-\frac{C_p}{C_v}(D+d) + \mathbf{S}^* D\right\}
\end{aligned} \tag{41}$$

after manipulation we obtain

$$\begin{aligned}
\frac{\partial^2 D}{\partial t^2} &= -gH^*\mathbf{G}^*\left\{-\kappa\mathbf{S}^*\triangle D + \epsilon\left[\kappa\mathbf{S}^*\triangle D - \tfrac{R}{C_v}\left(\triangle D + \triangle d\right)\right]\right\} \\
&\quad -gH^*\left\{-\mathbf{N}^*\triangle D\right\} \\
&\quad +gH^*(\mathbf{G}^* - 1)\epsilon\left\{-\tfrac{C_p}{C_v}(\triangle D + \triangle d) + \mathbf{S}^*\triangle D\right\}
\end{aligned} \tag{42}$$

After additional manipulation we obtain

$$\begin{aligned}
\frac{\partial^2 D}{\partial t^2} &= gH^*\kappa\mathbf{G}^*\mathbf{S}^*\triangle D + gH^*\mathbf{N}^*\triangle D \\
&\quad -gH^*\kappa\mathbf{G}^*\epsilon\mathbf{S}^*\triangle D \\
&\quad +gH^*\tfrac{R}{C_v}\mathbf{G}^*\epsilon\triangle D + gH^*\tfrac{R}{C_v}\mathbf{G}^*\epsilon\triangle d \\
&\quad -gH^*\tfrac{C_p}{C_v}\mathbf{G}^*\epsilon\triangle D + gH^*\tfrac{C_p}{C_v}\epsilon\triangle D \\
&\quad -gH^*\tfrac{C_p}{C_v}\mathbf{G}^*\epsilon\triangle d + gH^*\tfrac{C_p}{C_v}\epsilon\triangle d \\
&\quad +gH^*\mathbf{G}^*\epsilon\mathbf{S}^*\triangle D - gH^*\epsilon\mathbf{S}^*\triangle D.
\end{aligned} \tag{43}$$

First two terms are HY terms. Additional terms are NH departure.

$$\begin{aligned}
\frac{\partial^2 D}{\partial t^2} &= gH^*\kappa\mathbf{G}^*\mathbf{S}^*\triangle D + gH^*\mathbf{N}^*\triangle D \\
&\quad +gH^*\left(\tfrac{C_v}{C_p}\mathbf{G}^*\epsilon\mathbf{S}^* - \mathbf{G}^*\epsilon - \epsilon\mathbf{S}^*\right)\triangle D \\
&\quad +gH^*\left(\tfrac{R}{C_p}\mathbf{G}^*\epsilon\mathbf{S}^* - \tfrac{R}{C_p}\mathbf{G}^*\epsilon\mathbf{S}^* + \epsilon_s\mathbf{N}^* - \epsilon_s\mathbf{N}^*\right)\triangle D \\
&\quad +c^{*2}\epsilon\triangle D \\
&\quad +(-gH^*\mathbf{G}^* + c^{*2})\epsilon\triangle d
\end{aligned} \tag{44}$$

$\mathbf{A}_1^*$ constraint took the form

$$\mathbf{A}_1^* := \mathbf{G}^*\epsilon(\eta)\mathbf{S}^*X - \mathbf{G}^*\epsilon(\eta)X - \epsilon(\eta)\mathbf{S}^*X + \epsilon_s\mathbf{N}^*X = \mathbf{G}^*\frac{\pi^*}{m^*}\frac{\partial\epsilon(\eta)}{\partial\eta}\mathbf{S}^*X. \tag{45}$$

In order to employ previous relation we added red terms in [44]. After manipulations we obtain

$$\begin{aligned}
\frac{\partial^2 D}{\partial t^2} &= gH^*\left(\kappa\mathbf{G}^*\mathbf{S}^* + \mathbf{N}^*\right)\triangle D \\
&\quad +\left[gH^*\ \left(\mathbf{A}_1^* - \kappa\mathbf{G}^*\epsilon\mathbf{S}^* - \epsilon_s\mathbf{N}^*\right) + c^{*2}\epsilon\right]\triangle D \\
&\quad +(-gH^*\mathbf{G}^* + c^{*2})\epsilon\triangle d
\end{aligned} \tag{46}$$

The same procedure we repeat with equation for vertical divergence

$$\frac{\partial^2 d}{\partial t^2} = -\frac{g^2}{RT_a{}^*}\mathbf{V}_v^*\frac{\partial \hat{q}}{\partial t} \tag{47}$$

$$\frac{\partial^2 d}{\partial t^2} = \frac{1}{rH^{*2}}\mathbf{V}_v^*\left(-gH^*\mathbf{S}^* + c^{*2}\right)D + c^{*2}\frac{1}{rH^{*2}}\mathbf{V}_v^* d. \tag{48}$$

When system is discretized in time and in vertical direction, we can follow steps reported in Voitus report. Our implicit system to be solved in this version II. is

$$\begin{aligned}
q_s - \delta t\left(-\mathbf{N}^* m^2 D\right) &= q_{sRHS} \\
T - \delta t\left\{-\kappa T^*\mathbf{S}^* m^2 D + \epsilon\left[\kappa T^*\mathbf{S}^* m^2 D - \frac{RT^*}{C_v}\left(m^2 D + d\right)\right]\right\} &= T_{RHS} \\
D - \delta t\left[-R\mathbf{G}^*\triangle T - RT^*\triangle q_s - \triangle\phi_s + RT^*(\mathbf{G}^* - 1)\epsilon\triangle\hat{q}\right] &= D_{RHS} \\
d - \delta t\left(-\frac{g}{rH^*}\mathbf{V}_v^*\hat{q}\right) &= d_{RHS} \\
\hat{q} - \delta t\left[-\frac{C_p}{C_v}(m^2 D + d) + \mathbf{S}^* m^2 D\right] &= \hat{q}_{RHS}.
\end{aligned} \tag{49}$$

We define new operator

$$\mathbf{V}_v^{**}X = \frac{1}{rH^{*2}}\mathbf{V}_v^* X$$

We can rewrite our systemas

$$\begin{aligned}
\frac{\partial^2 D}{\partial t^2} &= \left[c^{*2}\mathbf{B}_{HY} + c^{*2}(\epsilon\mathbf{I} + \mathbf{C}_1 - \mathbf{B}_b)\right]\triangle D \\
&\quad + c^{*2}\mathbf{G}_\kappa^*\epsilon\triangle d \\
\frac{\partial^2 d}{\partial t^2} &= c^{*2}\mathbf{V}_v^{**}\mathbf{S}_\kappa^* D + c^{*2}\mathbf{V}_v^{**} d
\end{aligned} \tag{50}$$

With $\mathbf{C}_1 = -\frac{c_v}{c_p}\mathbf{A}_1^*$ and $\mathbf{B}_{HY} = \frac{c_v}{c_p}\left(\kappa\mathbf{G}^*\mathbf{S}^* + \mathbf{N}^*\right)$ and $\mathbf{B}_b = \frac{c_v}{c_p}\left(\kappa\mathbf{G}^*\epsilon\mathbf{S}^* + \epsilon_s\mathbf{N}^*\right)$. We have to realize that $\mathbf{C}_1$ in our case is discretized version of

$$\mathbf{C}_1 = \frac{C_v}{C_p}\left[\mathbf{G}^*\epsilon\mathbf{S}^* X - \mathbf{G}^*\epsilon X - \epsilon\mathbf{S}^* X + \epsilon_s\mathbf{N}^* X\right].$$

When we assume also that the system is discretized in time we obtain equivalent of equations (9) and (10) from report of Voitus,

$$\begin{aligned}
\left[\mathbf{I} - \delta t^2 c^{*2}\mathbf{B}_{HY} - \delta t^2 c^{*2}(\epsilon\mathbf{I} + \mathbf{C}_1 - \mathbf{B}_b)\right] m^2\triangle D \\
-\delta t^2 c^{*2}\mathbf{G}_\kappa^*\epsilon\triangle d &= D^\bullet \\
\left[\mathbf{I} - \delta t^2 c^{*2}\mathbf{V}_v^{**}\right] d - \delta t^2 c^{*2}\mathbf{V}_v^{**}\mathbf{S}_\kappa^* D &= d^\bullet
\end{aligned} \tag{51}$$

with (11) and (12) of Voitus being modified as

$$D^{\bullet} = D_{RHS} - \delta t R T^* \triangle \left[(\mathbf{I} - \mathbf{G}^*) \epsilon \hat{q}_{RHS} + \mathbf{G}^* \frac{T_{RHS}}{T^*} + q_{sRHS} \right] \tag{52}$$

and

$$d^{\bullet} = d_{RHS} - \delta t R T^* \mathbf{V}_v^{**} \hat{q}_{RHS}. \tag{53}$$

When we define now $\mathbf{H}_v^*$ as

$$\mathbf{H}_v^* = \mathbf{I} - \delta t^2 c^{*2} \mathbf{V}_v^{**}$$

and we modify (14) from Voitus as

$$d = \mathbf{H}_v^{*-1} \left[d^{\bullet} + \delta t^2 c^{*2} \mathbf{V}_v^{**} \mathbf{S}_\kappa^* D \right]. \tag{54}$$

then we obtain

$$\begin{aligned} \left[\mathbf{I} - \delta t^2 c^{*2} \mathbf{B}_{HY} - \delta t^2 c^{*2} \triangle (\epsilon \mathbf{I} + \mathbf{C}_1 - \mathbf{B}_b) \right] m^2 D &= \\ D^{\bullet} + \delta t^2 c^{*2} \triangle \mathbf{G}_\kappa^* \epsilon \mathbf{H}_v^{*-1} \left[d^{\bullet} + \delta t^2 c^{*2} \mathbf{V}_v^{**} \mathbf{S}_\kappa^* D \right] &= \\ D^{\bullet\bullet} + \delta t^4 c^{*4} \triangle \mathbf{G}_\kappa^* \epsilon \mathbf{H}_v^{*-1} \mathbf{V}_v^{**} \mathbf{S}_\kappa^* m^2 D & \end{aligned} \tag{55}$$

with

$$D^{\bullet\bullet} = D^{\bullet} + \delta t^2 c^{*2} \triangle \mathbf{G}_\kappa^* \epsilon \mathbf{H}_v^{*-1} d^{\bullet}. \tag{56}$$

Resulting Helmholtz equation then yields

$$\left[1 - \delta t^2 \mathbf{B} \triangle \right] D = D^{\bullet\bullet} \tag{57}$$

with

$$\mathbf{B} = c^{*2} \left[\mathbf{B}_{HY} + \left(\epsilon \mathbf{I} + \mathbf{C}_1 - \mathbf{B}_b + \delta t^2 c^{*2} \mathbf{G}_\kappa^* \epsilon \mathbf{H}_v^{*-1} \mathbf{V}_v^{**} \mathbf{S}_\kappa^* \right) \right]. \tag{58}$$

It can be shown that NH part can be rewritten as

$$
\begin{aligned}
\epsilon\mathbf{I} + \mathbf{C}_1 - \mathbf{B}_b + \delta t^2 c^{*2}\mathbf{G}^*_\kappa\epsilon\mathbf{H}^{*-1}_v\mathbf{V}^{**}_v\epsilon\mathbf{S}^*_\kappa &= \\
\epsilon\mathbf{I} + \mathbf{C}_1 - \mathbf{B}_b + \mathbf{G}^*_\kappa\epsilon\mathbf{H}^{*-1}_v(1-\mathbf{H}^*_v)\mathbf{S}^*_\kappa &= \\
\epsilon\mathbf{I} + \mathbf{C}_1 - \mathbf{B}_b + \mathbf{G}^*_\kappa\epsilon\mathbf{H}^{*-1}_v\mathbf{S}^*_\kappa - \mathbf{G}^*_\kappa\epsilon\mathbf{S}^*_\kappa &= \\
\epsilon\mathbf{I} + \tfrac{C_v}{C_p}\mathbf{G}^*\epsilon\mathbf{S}^* - \tfrac{C_v}{C_p}\mathbf{G}^*\epsilon - \tfrac{C_v}{C_p}\epsilon\mathbf{S}^* + \tfrac{C_v}{C_p}\epsilon_s\mathbf{N}^* & \\
-\tfrac{c_v}{c_p}\kappa\mathbf{G}^*\epsilon\mathbf{S}^* - \tfrac{c_v}{c_p}\epsilon_s\mathbf{N}^* & \\
-\epsilon\mathbf{I} + \tfrac{C_v}{C_p}\epsilon\mathbf{S}^* + \tfrac{C_v}{C_p}\mathbf{G}^*\epsilon - \left(\tfrac{C_v}{C_p}\right)^2\mathbf{G}^*\epsilon\mathbf{S}^* & \\
+\mathbf{G}^*_\kappa\epsilon\mathbf{H}^{*-1}_v\mathbf{S}^*_\kappa &= \\
\tfrac{C_v}{C_p}\left(1-\kappa-\tfrac{C_v}{C_p}\right)\mathbf{G}^*\epsilon\mathbf{S}^* + \mathbf{G}^*_\kappa\epsilon\mathbf{H}^{*-1}_v\mathbf{S}^*_\kappa &= \mathbf{G}^*_\kappa\epsilon\mathbf{H}^{*-1}_v\mathbf{S}^*_\kappa
\end{aligned}
\tag{59}
$$

when using $\delta t^2 c^{*2}\mathbf{V}^{**}_v = \mathbf{I} - \mathbf{H}^*_v$ and $1-\kappa-\frac{C_v}{C_p} = 0$.

After re-arrangment we can show finally that

$$\mathbf{B} = c^{*2}\left(\mathbf{B}_{HY} + \mathbf{B}_{NH}\right). \tag{60}$$

with

$$B_{NH} = \mathbf{G}^*_\kappa\epsilon\mathbf{H}^{*-1}_v\mathbf{S}^*_\kappa. \tag{61}$$

5 Implementation (version I.)

In order to better understand SI computation here we conclude what is computed by SI routines.

$$
\begin{aligned}
SITNU_{PT}(PD) &= \kappa T^*\mathbf{S}^*(PD) \\
SITNU_{PSP}(PD) &= \mathbf{N}^*(PD) \\
SIGAM_{PD}(PT, PSP) &= R\mathbf{G}^*(PT) + RT^*(PSP) \\
SISEVE_{PV2}(PV1) &= \tfrac{1}{r}\mathbf{L}^*_v(PV1) \\
SIDD_{PDH}(PT, PSP, PRNH) &= R\mathbf{G}^*(PT) + RT^*(PSP) + RT^*(1-\mathbf{G}^*)\epsilon(PRNH) \\
SIDD_{PDV}(PRNH) &= \tfrac{g^2}{RT^*}\tfrac{1}{r}\mathbf{L}^*_v(PRNH) \\
SIPTP_{PT}(PDH, PDV) &= \tfrac{RT^*}{C_v}(PDH + PDV) \\
SIPTP_{PRNH}(PDH, PDV) &= \tfrac{C_p}{C_v}(PDH + PDV) - \mathbf{S}^*(PDH)
\end{aligned}
\tag{62}
$$

with $r = \frac{T_a^*}{T^*} = \frac{SITRA}{SITR}$ applied by default (can be off via optional argument).

Basic SI matrices are implemented in routine SUNHEEBMAT as follows

$$
\begin{aligned}
ZSIB\_HYD &= c^{*2}\tfrac{C_v}{C_p}\left(\kappa\mathbf{G}^*\mathbf{S}^* + \mathbf{N}^*\right) \\
SIFACI &= \left(\mathbf{I} - \delta t^2 c^{*2}\mathbf{L}_v^{**}\epsilon\right)^{-1} = \mathbf{H}_v^{*-1}
\end{aligned} \tag{63}
$$

we use relation $RT^* = RT^*\frac{C_p}{C_v}\frac{C_v}{C_p} = c^{*2}\frac{C_v}{C_p}$.

Taking into account that NH addictional part shall be $c^{*2}\mathbf{G}_\kappa^*\mathbf{H}_v^{*-1}\mathbf{S}_\kappa^*$ we were surprised by the length of original code in CY47 in routine SUNHEEB-MAT. Ee analyzed the computation and here they are step by step

$$
\begin{aligned}
ZZ1 &= c^{*2}\mathbf{I} - C_p\kappa T^*\mathbf{S}^* = c^{*2}\mathbf{S}_\kappa^* \\
ZZ21 &= c^{*2}\tfrac{1}{r}\mathbf{L}_v^*\mathbf{S}_\kappa^* \\
ZZ22 &= c^{*2}\tfrac{1}{r}\mathbf{H}_v^{*-1}\mathbf{L}_v^*\mathbf{S}_\kappa^* \\
ZZ2 &= c^{*2}\left(\delta t^2 c^{*2}\mathbf{H}_v^{*-1}\mathbf{L}_v^{**} + \mathbf{I}\right)\mathbf{S}_\kappa^* \\
ZZ3 &= Rc^{*2}\mathbf{G}^*\left(\delta t^2 c^{*2}\mathbf{H}_v^{*-1}\mathbf{L}_v^{**} + \mathbf{I}\right)\mathbf{S}_\kappa^* \\
ZSIB\_ADD &= \left(\mathbf{I} - \tfrac{C_v}{C_p}\mathbf{G}^*\right)c^{*2}\left(\delta t^2 c^{*2}\mathbf{H}_v^{*-1}\mathbf{L}_v^{**} + \mathbf{I}\right)\mathbf{S}_\kappa^* \\
&= c^{*2}\mathbf{G}_\kappa^*\left(\delta t^2 c^{*2}\mathbf{H}_v^{*-1}\mathbf{L}_v^{**} + \mathbf{I}\right)\mathbf{S}_\kappa^* \\
&= c^{*2}\mathbf{G}_\kappa^*\left[\mathbf{H}_v^{*-1}\left(\mathbf{I} - \mathbf{H}_v^*\right) + \mathbf{I}\right]\mathbf{S}_\kappa^* \\
&= c^{*2}\mathbf{G}_\kappa^*\mathbf{H}_v^{*-1}\mathbf{S}_\kappa^*
\end{aligned} \tag{64}
$$

We considered this implementation to be very complicated for understanding. Therefore we have implemented following simpler version without any need to compute vertical laplacian operator $\mathbf{L}_v^*$

$$
\begin{aligned}
ZZ1 &= c^{*2}\mathbf{I} - C_p\kappa T^*\mathbf{S}^* = c^{*2}\mathbf{S}_\kappa^* \\
ZZ2 &= \epsilon^2\mathbf{H}_v^{*-1}c^{*2}\mathbf{S}_\kappa^* \\
ZZ3 &= R\mathbf{G}^*\epsilon^2\mathbf{H}_v^{*-1}c^{*2}\mathbf{S}_\kappa^* \\
ZSIB\_ADD &= (\mathbf{I} - \tfrac{C_v}{C_pR}R\mathbf{G}^*)\epsilon^2\mathbf{H}_v^{*-1}c^{*2}\mathbf{S}_\kappa^* \\
&= c^{*2}\mathbf{G}_\kappa^*\epsilon^2\mathbf{H}_v^{*-1}\mathbf{S}_\kappa^*
\end{aligned} \tag{65}
$$

We perform validation. The new simpler implementation sligtly differ at the last digit of spectral norms.

Next step was implementation of ϵ into spectral space computations into routine ESPNHEESI. Following quantities were modified consistently with this report

$$
\begin{aligned}
ZSDIV &= R\mathbf{G}^* T_{RHS} + RT^* q_{sRHS} + RT^*(\mathbf{I}-\mathbf{G}^*)\epsilon\hat{q}_{RHS} \\
ZSVED &= \frac{g^2}{RT^*}\frac{1}{r}\mathbf{L}_v^*\hat{q}_{RHS} \\
ZDH\_DOT &= D_{RHS} - \delta t\triangle R\mathbf{G}^* T_{RHS} + RT^* q_{sRHS} + RT^*(\mathbf{I}-\mathbf{G}^*)\epsilon\hat{q}_{RHS} = D^\bullet \\
ZVD\_DOT &= \mathbf{I} - \delta t\frac{g^2}{RT^*}\frac{1}{r}\mathbf{L}_v^*\hat{q}_{RHS} = d^\bullet \\
Z11 &= \epsilon^2\mathbf{H}_v^{*-1} d^\bullet \\
Z12 &= R\mathbf{G}^*\epsilon^2\mathbf{H}_v^{*-1} d^\bullet \\
ZSRHS &= D^\bullet - \delta t^2\triangle\left(T^* R\mathbf{G}^*\epsilon^2\mathbf{H}_v^{*-1} d^\bullet - c^{*2}\epsilon^2\mathbf{H}_v^{*-1} d^\bullet\right) \\
&= D^\bullet + \delta t^2\triangle c^{*2}\mathbf{G}_\kappa^*\epsilon^2\mathbf{H}_v^{*-1} d^\bullet = D^{\bullet\bullet} \\
ZZSPDIVG &= \left(\mathbf{I} - \delta t^2\triangle\mathbf{B}\right)^{-1} D^{\bullet\bullet}
\end{aligned}
\tag{66}
$$

The solution of ZZSPDIVG is performed in eigenspace of vertical eigenmodes. Then we compute spectral coefficients of d (ZZSPDIVG) via following steps

$$
\begin{aligned}
Z21 &= \epsilon C_p\kappa T^*\mathbf{S}^* m^2 D - c^{*2} m^2 D = -c^{*2}\epsilon\mathbf{S}_\kappa^* m^2 D \\
Z22 &= -\frac{1}{r}\mathbf{L}_v^* c^{*2}\epsilon\mathbf{S}_\kappa^* m^2 D \\
Z23 &= d^\bullet + \frac{\delta t^2}{H^{*2}}\frac{1}{r}\mathbf{L}_v^* c^{*2}\epsilon\mathbf{S}_\kappa^* m^2 D \\
&= d^\bullet + \delta t^2 c^{*2}\mathbf{L}_v^{**}\epsilon\mathbf{S}_\kappa^* m^2 D \\
ZZSPSVDG &= \mathbf{H}_v^{*-1}\left(d^\bullet + \delta t^2 c^{*2}\mathbf{L}_v^{**}\epsilon\mathbf{S}_\kappa^* m^2 D\right)
\end{aligned}
\tag{67}
$$

We compute new value of $\hat{q}$ (ZZSPSPDG)

$$
\begin{aligned}
ZZSPSPDG &= \hat{q}_{RHS} - \delta t\epsilon\left[\frac{C_p}{C_v}\left(m^2 D + d\right) - \frac{C_p}{RT^*}\kappa T^*\mathbf{S}^* m^2 D\right] \\
&= \hat{q}_{RHS} - \delta t\epsilon\left[\frac{C_p}{C_v}\left(m^2 D + d\right) - \mathbf{S}^* m^2 D\right]
\end{aligned}
\tag{68}
$$

We compute new value of T (ZZSPTG)

$$
ZZSPTG = T_{RHS} - \delta t\epsilon^2\frac{RT^*}{C_v}\left(m^2 D + d\right) - \delta t\left(1-\epsilon^2\right)\kappa T^*\mathbf{S}^* m^2 D.
\tag{69}
$$

Further we modifies grid point part of SI computation (only SL part) inside LANHSI routine.

Nonlinear model is modified in routines CPG_GP_NHEE. First we modifify computation of geopotential

$$ZRRED0 \quad = \quad R * \left[1 + \epsilon \left(\frac{\pi}{p} - 1\right)\right] \tag{70}$$

then in routine GPGEO we compute geopotential as before. Consistenly we modified also calculations of gradient of geopotential in GPGRGEO.

6 Tests with constant ϵ

7 Tests with vertically varying ϵ

8 Conclusion

Previous formulation allow us to control non-hydrostaticity of model. This has no meaning in full non-linear model as only solution with $\epsilon = 0$ or $\epsilon = 1$ has physical meaning. However it could be worth to investigate the stability of NH model with $\epsilon > 1$ (over-non-hydrostatic). I will study this.