New vertical motion variables in the non-hydrostatic dynamical core of the ALADIN system

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***Abstract*—Motivated by the work of Fabrice Voitus (see his presentation from the ALADIN Workshop in Toulouse 2018), we have implemented vertical prognostic variables d_5 and gW. The time stepping of gW quantity includes Y-term. The time stepping implementation of the Y-term is equivalent to the X-term implementation (Smolikova report). To better understand the stability properties of the Y-term treatment we performed a stability analysis based on Gospodinov and Smolikova. The scheme reported here was developed and tested exclusively for the following switches LGWADV=.T., ND4SYS=2, NXLAG=3, LTWOTL=.T. and LSLAG=.T..**

I. New prognostic quantities

Fabrice showed that the robustness of existing constant coefficient SI SL scheme can be improved. The key is different redistribution of D_3 in between prognostic quantities. Actually we are using division

$$D_3 = D - \frac{p}{mRT}\frac{\partial gw}{\partial \eta} + \frac{p}{mRT}\frac{\partial \vec{v}}{\partial \eta}\vec{\nabla}\phi, \tag{1}$$

with prognostic quantity that satisfies $D_3 = D + d_4$. This gives

$$d_4 = -\frac{p}{mRT}\frac{\partial gw}{\partial \eta} + X4 \tag{2}$$

$$X4 = \frac{p}{mRT}\frac{\partial \vec{v}}{\partial \eta}\vec{\nabla}\phi \tag{3}$$

where

$$gw = \frac{d\phi}{dt}, \tag{4}$$

with the free slip surface boundary condition

$$gw_s = \vec{v}_s\vec{\nabla}\phi_s. \tag{5}$$

The new division is based on the idea that the new vertical "velocity" quantity written with capital W will have zero value at the surface

$$gW_s = 0. \tag{6}$$

We use two different divisions

1) LVD5W $(W5, d_5, X5, Y5)$

$$gW5 = gw - \vec{v}\vec{\nabla}\phi = gw + Y5. \tag{7}$$

The Y-term is defined in (7). The new division of D_3 with $gW5$ used is

$$D_3 = D - \frac{p}{mRT}\frac{\partial gW5}{\partial \eta} - \frac{p}{mRT}\vec{v}\vec{\nabla}\frac{\partial \phi}{\partial \eta}. \tag{8}$$

We define new prognostic quantities

$$d_5 = -\frac{p}{mRT}\frac{\partial gW5}{\partial \eta} + X5 \tag{9}$$

$$X5 = -\frac{p}{mRT}\vec{v}\vec{\nabla}\frac{\partial \phi}{\partial \eta} \tag{10}$$

2) LVD6W $(W6, d_6, X6, Y6)$

$$gW6 = gw - \vec{v}\vec{\nabla}\phi_s = gw + Y6. \tag{11}$$

The new division of D_3 is

$$D_3 = D - \frac{p}{mRT}\frac{\partial gW6}{\partial \eta} + X4 - \frac{p}{mRT}\frac{\partial \vec{v}\vec{\nabla}\phi_s}{\partial \eta}. \tag{12}$$

We define new prognostic quantities

$$d_6 = -\frac{p}{mRT}\frac{\partial gW6}{\partial \eta} + X6 \tag{13}$$

$$X6 = X4 - \frac{p}{mRT}\frac{\partial \vec{v}\vec{\nabla}\phi_s}{\partial \eta}. \tag{14}$$

The evolution of quantities is equivalent to actual implementation of $(d_4, X4, gw)$. We compute evolution of gw resp. gW and we transform explicit guess of prognostic quantities into d variable. We than add SI linear correction evaluated for d. The spectral computations are performed with d being prognostic. This trick requires special treatment of the X-term and the Y-term. We use W, d, X, Y without specifying which of the two divisions is used if there is no danger of confusion. Always W, d, X, Y used in one equation correspond to each other.

From the equation

$$D - \frac{p}{mRT}\frac{\partial gw + Y}{\partial \eta} + X = D - \frac{p}{mRT}\frac{\partial gw}{\partial \eta} + X4 \tag{15}$$

follows that we can compute the Y-term using the following relations

$$\frac{p}{mRT}\frac{\partial Y}{\partial \eta} = X - X4, \frac{\partial \phi}{\partial \eta} = -\frac{mRT}{p} \tag{16}$$

as

$$\frac{\partial Y}{\partial \phi} = X4 - X. \tag{17}$$

II. Evolution of gW and d variables

Here we introduce special treatment of the Y-term. We assume that Y can be divided into two parts $Y = Y_e + Y_{sl}$. We know exactly the nonlinear model of Y_e time evolution. The Y_{sl} model is unknown (or too complex) and therefore we solve its evolution by SL numerical approximation of $\frac{dY_{sl}}{dt}$ term. The evolution of gW yields

$$\frac{dgW}{dt} = N + \frac{dY_{sl}}{dt} \tag{18}$$

and nonlinear model N contains terms related to evolution of gw and Y_e. The division of Y is as follows

variable	Y	Y_e	Y_{sl}
$Y5$	$-\vec{v}\vec{\nabla}\phi$	0	$-\vec{v}\vec{\nabla}\phi$
$Y6$	$-\vec{v}\vec{\nabla}\phi_s$	$-\left(\vec{v}-\vec{v_s}\right)\vec{\nabla}\phi_s$	$-\vec{v_s}\vec{\nabla}\phi_s$

and the time evolution yields

variable	$\frac{dY_e}{dt}$	$\frac{dY_{sl}}{dt}$
$Y5$	0	$-\frac{d(\vec{v}\vec{\nabla}\phi)}{dt}$
$Y6$	$-\frac{d(\vec{v}-\vec{v_s})}{dt}\vec{\nabla}\phi_s$	$-\frac{d(\vec{v_s}\vec{\nabla}\phi_s)}{dt}$

Remark: The simplest choice $Y_{sl} = 0$ for $Y6$ leads to the development of the so called ”chimney” over the orographic obstacle as described in (Brozkova, Smolikova). We solve this problem by adding the time evolution of $\vec{v_s}\vec{\nabla}\phi_s$ into the nonlinear model N. We have to compensate the time evolution of $\vec{v_s}\vec{\nabla}\phi_s$ included in the nonlinear model N through the SL-treatment of $-\vec{v_s}\vec{\nabla}\phi_s$.

Variable gW is a half-level quantity (as gw) and therefore also the Y-term is a half level quantity. The explicit guess of gW is computed using 2TL SI SL scheme as

$$\frac{gW_F^+ - gW_O^0}{dt} = N_M^m + \frac{Y_{slF}^+ - Y_{slO}^0}{dt}. \tag{19}$$

In order to write the time scheme with the SL advection we introduce the following notation

symbol	value	meaning
O	$x - dx$	the origin (departure) point of the trajectory
F	x	the final (arrival) point of the trajectory
M	$x - \frac{dx}{2}$	the middle point of the trajectory
$+$	$t + dt$	
0	t	
m	$t - \frac{dt}{2}$	
$-$	$t - dt$	

The terms Y_{slF}^+ and N_M^m are extrapolated using values at time levels t and $t - dt$ and spatial locations F and O with second order method in time. This gives one parametric relations

$$\begin{aligned} Y_{slF}^+ &= (\frac{3}{2}-\alpha)Y_{slF}^0 + (\alpha - \frac{1}{2})Y_{slF}^- \\ &\quad +(\alpha+\frac{1}{2})Y_{slO}^0 + (-\alpha-\frac{1}{2})Y_{slO}^- + O(dt^2). \end{aligned} \tag{20}$$

and

$$N_M^m = (\frac{3}{4}-\beta)N_F^0 + (\beta-\frac{1}{4})N_F^- \tag{21}$$

$$+(\beta+\frac{3}{4})N_O^0 + (-\beta-\frac{1}{4})N_O^- + O(dt^2). \tag{22}$$

A stability analysis of 19 depending on parameters α and β is given in section VII.

The choice $\alpha = 1/2$ leads to the time stepping treatment of Y_e equivalent to the time treatment of the X-term under $ND4SYS = 2$. This gives

$$\frac{gW_F^+ - gW_O^0}{dt} = N_M^m + \frac{Y_{slF}^0 - Y_{slO}^-}{dt}. \tag{23}$$

When SETTLS ($\beta = \frac{1}{4}$) is used the time stepping is second order accurate.

The treatment of Y_{sl} requires an addition of the new terms at the level of LATTEX_DNT. Division between SL buffers is as follows ($\tau = \frac{dt}{2}$)

$$\begin{aligned} &PredictorNESC \\ gW_F^+ &= \left[gW^0 + \tau N^0 - Y_{sl}^-\right]_O \\ &\quad + \left[\tau N^0 + Y_{sl}^0\right]_F \end{aligned} \tag{24}$$

$$\begin{aligned} &PredictorSETTLS \\ gW_F^+ &= \left[gW^0 + 2\tau N^0 - \tau N^- - Y_{sl}^-\right]_O \\ &\quad + \left[\tau N^0 + Y_{sl}^0\right]_F \end{aligned} \tag{25}$$

$$\begin{aligned} &Corrector \\ gW_F^{+(n)} &= \left[gW^0 + \tau N^0 - Y_{sl}^0\right]_O \\ &\quad + \left[\tau N^{+(n-1)} + Y_{sl}^{+(n-1)}\right]_F \end{aligned} \tag{26}$$

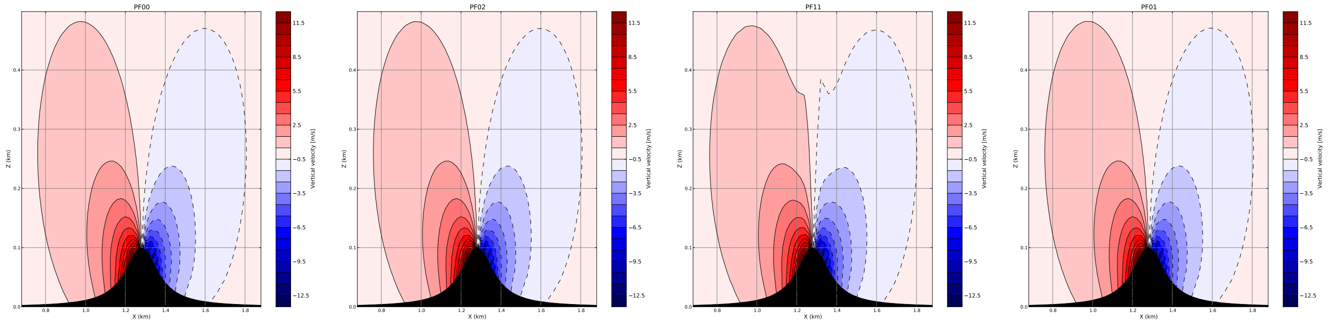
The results are shown in Figure 1. The ”chimney” effect is apparent in (c). The SL approximative treatment of BBC is shown in (d). One can see that the problem has disappeared.

III. Linear model for d_5 and d_6

In the continuous context

$$d_4 = d_5 = d_6 \tag{27}$$

and therefore the prognostic equation for d_5 and d_6 must be the same as for d_4. The vertical divergence d is a prognostic variable in the linear computations treated in the SI manner. Hence the linear model remains unchanged for all d/w choices while the shape of the nonlinear model formulated with vertical velocity w variable (LGWADV=TRUE) depends on the w variable choice.



(a) reference (b) LVD5W (c) LVD6W, $Y_{sl} = 0$ (d) LVD6W, $Y_{sl} = \vec{v_s}\vec{\nabla}\phi_s$

Figure 1: The potential flow test for new d and W variables.

IV. Vertical discretisation

To simplify the notation we introduce the depth of the model layer

$$d\phi_l = \left(\frac{\pi}{p}\right)_l (RT)_l \left(\frac{d\pi}{\pi}\right)_l . \tag{28}$$

The $X4$-term is discretized on the model full levels (GPXX routine) as

$$X4_l = \frac{\left(\vec{v}_{\tilde{l}} - \vec{v}_l\right)\vec{\nabla}\phi_{\tilde{l}} + \left(\vec{v}_l - \vec{v}_{\tilde{l}-1}\right)\vec{\nabla}\phi_{\tilde{l}-1}}{d\phi_l}. \tag{29}$$

The $X5$-term is discretized in an analogous way as

$$X5_l = -\frac{\left(\vec{\nabla}\phi_{\tilde{l}} - \vec{\nabla}\phi_l\right)\vec{v}_{\tilde{l}} + \left(\vec{\nabla}\phi_l - \vec{\nabla}\phi_{\tilde{l}-1}\right)\vec{v}_{\tilde{l}-1}}{d\phi_l}. \tag{30}$$

The $X6$-term is discretized as

$$X6_l = X4_l - \frac{\left(\vec{v}_{\tilde{l}} - \vec{v}_{\tilde{l}-1}\right)}{d\phi_l}\vec{\nabla}\phi_s. \tag{31}$$

V. Conversion from model to file and vice versa

There is a full level quantity $-gdw$ stored in FA files. This quantity is being converted into the vertical divergence d (this and backward conversion is calculated in GNH_CONV_NHVAR). The full level quantity dY is used in this conversion. It is computed from the relevant X-terms using (17) as

$$dY_l = \left(X4_l - X_l\right) d\phi_l. \tag{32}$$

The half level quantity Y itself is computed by vertical integration of dY using an appropriate bottom boundary condition

$$Y_s = Y_{\tilde{L}} = -\vec{v}_L\vec{\nabla}\phi_{\tilde{L}} \tag{33}$$

and

$$Y_{\tilde{l}-1} = Y_{\tilde{l}} - dY_l. \tag{34}$$

VI. Idealized and real experiments

We tested all three schemes (SI LSETTLS, SI NESC, PC NESC CHEAP) with the potential flow test. We found big sensitivity of the time stepping stability to the time treatment of Y_{sl}. The SI SETTLS scheme is the most stable one. The SI NESC scheme stability is achieved only when $\frac{\partial Y_{sl}}{\partial \eta} = 0$ at the model top level. If this is not fulfilled the scheme requires a sponge application in order to be stable. The PC scheme with NESC predictor is unstable. This problem was not investigated in details, it was left for further work. A general conclusion from idealized experiments is that the best choice is SI SETTLS scheme with the sponge activated.

It is also important fact that stability of the full 3D model is not consistent with the results of the stability analysis presented in section VII where $\alpha = \frac{1}{2}$ is not favorable. Experiments show stability exclusively for $\alpha = \frac{1}{2}$. Any other value leads to unstable SI scheme, despite the fact that the stability analysis results are more optimistic.

VII. Stability analysis of the SL scheme

We extend the analysis of stability from Gospodinov and Vivoda. We assume that prognostic quantity $f(x,t)$ can be linearly divided using a parameter δ as $f = \delta f + (1-\delta) f$ and the time evolution of f is described by

$$\frac{df}{dt} = \delta\left(\lambda + i\omega\right) f + (1-\delta)\frac{df}{dt}. \tag{35}$$

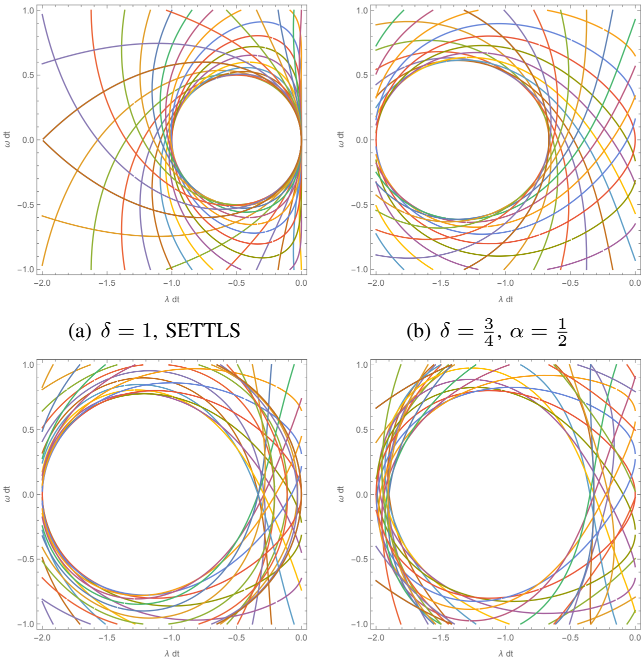
The right hand side of a prognostic equation is thus split in an analytically expressed part and a part which evolution is treated approximatively in SL-manner due to its very nonlinear nature. This is a prototype of equations for d_4, resp. d_5, and gW in the NH kernel of the ALADIN system. The part $(1-\delta)\frac{df}{dt}$ represents the time evolution of the X-term, Y-term respectively. The 2TL time discretization gives

$$f_F^+ - f_O^0 = \delta\left(dt\lambda + idt\omega\right) f_M^m + (1-\delta)\left(\tilde{f}_F^+ - f_O^0\right) \tag{36}$$

The terms $f_M^m = f(-\frac{dx}{2}, \frac{dt}{2})$ and $\tilde{f}_F^+ = f(0, dt)$ are extrapolated using values at time levels 0 and $-$ and spatial locations F and O with second order method in time. This gives

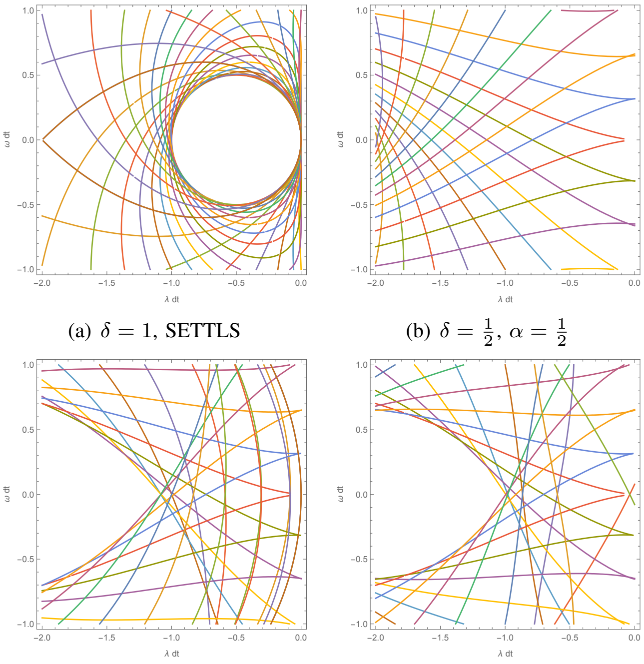
$$\begin{aligned} f_M^m &= (\frac{3}{4} - \beta) f_F^0 + (\beta - \frac{1}{4}) f_F^- \\ &\quad + (\beta + \frac{3}{4}) f_O^0 + (-\beta - \frac{1}{4}) f_O^- + O(dt^2). \end{aligned} \tag{37}$$

$$\begin{aligned} \tilde{f}_F^+ &= (\frac{3}{2} - \alpha) f_F^0 + (\alpha - \frac{1}{2}) f_F^- \\ &\quad + (\alpha + \frac{1}{2}) f_O^0 + (-\alpha - \frac{1}{2}) f_O^- + O(dt^2).. \end{aligned} \tag{38}$$



(a) $\delta = 1$, SETTLS (b) $\delta = \frac{3}{4}$, $\alpha = \frac{1}{2}$ (c) $\delta = \frac{3}{4}$, $\alpha = 0$ (d) $\delta = \frac{3}{4}$, $\alpha = -\frac{1}{2}$

Figure 2: Stability of SL time stepping for various values of α.



(a) $\delta = 1$, SETTLS (b) $\delta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ (c) $\delta = \frac{1}{2}$, $\alpha = 0$ (d) $\delta = \frac{1}{2}$, $\alpha = -\frac{1}{2}$

Figure 3: Stability of SL time stepping for various values of α.

The scheme is second order accurate in time for any choice of parameters α and β. However, its stability depends on the choice of these parameters. We investigate solely the value $\beta = \frac{1}{4}$ that represents the SETTLS scheme. The $ND4SYS = 2$ solution of the X-term treatment is represented by the choice $\alpha = \frac{1}{2}$.

The stability results for the SETTLS scheme with relatively big contribution of the explicit term ($\delta = \frac{3}{4}$) are shown in Figure 2 for a range of values $(dt\lambda, dt\omega)$. The stability is enhanced when compared to the stability of the reference SETTLS scheme (a).

The stability results for the SETTLS scheme with relatively smaller contribution of the explicit term ($\delta = \frac{1}{2}$) are shown in Figure 3. Again the SETTLS reference is shown as (a).

We showed that the stability of the X-term and the Y-term time stepping can be improved while keeping second order of accuracy in time. However as we showed the relative contribution of approximative time evolution term must be small when compared to the analytic one ($1 - \delta << \delta$).

As mentioned already above the presented analysis is based on Hortal resp. Gospodinov, where such an analysis is used to prove the stability properties of the SETTLS scheme. However, we found that the results from 3D experiments are not consistent with the theoretical results of this analysis. Therefore we are skeptical also about the results obtained for SL scheme without any approximative SL term. The full 3D model seems to be too complex to be approximated by a simple 1D single variable equation.

VIII. Implementation into model

The newly proposed d_5, d_6 and $W5, W6$ were implemented into the CY46 of the ALADIN system under the key LVD5W, LVD6W respectively, exclusively under the following choices

key	value	meaning
ND4SYS	2	treatment of X variable (can be improved !)
LGWADV	T	using vertical velocity as prognostic quantity for SL advection and nonlinear evolution
LTWOTL	T	only 2TL scheme considered
LSLAG	T	only semi-Lagrangian scheme considered
NXLAG	3	Y term evolution coded only for this choice
NVDVAR	4	new variables are implemented as small deviation from this choice

The following new keys were introduced in the new module YOMDEV and are supposed to be moved to YOMDYN:

key	value	meaning	default
LVD5W	T	NEW key used to activate d_5 and $gW5$ computations	F
LVD6W	T	NEW key used to activate d_6 and $gW6$ computations	F
LVDWY	T	use the SL approximation of $\frac{\partial Y}{\partial t}$ evolution set if LVD5W=T	F
	F	use the analytical form to evolve Y with LVD6W=T only	
LPGFH	T	use the PGF half level quantity in Y calculations	F
	F	use half level horizontal wind tendency in Y calculations with LVD6W=T only	
NX5FORM	integer	the way $X5$ is discretized (in GPXX)	2
RWY	real	α = a weight to control Y time discretisation RWY = 1/2 $\approx$ ND4SYS = 2	$\frac{1}{2}$