Vertical finite element scheme in dynamical core of ALADIN

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I. Introduction

The stay has been dedicated mainly to finish paper about previous work. The first draft of paper has been written already 2 years ago, but meanwhile the code diverge from waht was written in the draft. Therefore we decided to revisit paper, to re-create 2D and 3D test with the newest version of the code.

We phased latest development into CY45, as a preparation for the future CY45T1. The version phased was the one at the end of last stay (June 2017). We also re-phased the same code into CY43T2BF02 because this version of model has been available at very moment at servers at CHMI and SHMI as well.

There was slight inconsistency between my latest locally developed code and what was phased into CY45 and CY43T2BF02. Therefore I put my modifications (differences in 6 routines only) on top of CY43T2BF02. We agreed that these modifications will enter CY45 in order to ensure their presence in main trunk in Toulouse.

Here we describe changes phased by me in top of CY43T2BF02 and we put here also 2D tests prepared for article.

II. Definition of full level A and B and half and full level depths δA and δB

Vertical discretization in FD scheme is based on implicit definition of half level hydrostatic pressures

$$\pi_{\tilde{l}} \quad = \quad A_{\tilde{l}} + B_{\tilde{l}}\pi_s. \tag{1}$$

The half-level values $A_{\tilde{l}}$ and $B_{\tilde{l}}$ are specified a priori and the values at domain top are $A_{\tilde{0}} = B_{\tilde{0}} = 0$ and at surface $A_{\tilde{L}} = 0$ and $B_{\tilde{L}} = 1$.

The FE scheme discretization is based on derivate form of (1)

$$\frac{\delta\pi_l}{\delta\eta_l} \quad = \quad \frac{\delta A_l}{\delta\eta_l} + \frac{\delta B_l}{\delta\eta_l}\pi_s \tag{2}$$

and conditions

$$\underline{\mathbf{I_0^1}} \cdot \frac{\boldsymbol{\delta A}}{\boldsymbol{\delta\eta}} \quad = \quad 0 \tag{3}$$

$$\underline{\mathbf{I_0^1}} \cdot \frac{\boldsymbol{\delta B}}{\boldsymbol{\delta\eta}} \quad = \quad 1 \tag{4}$$

with all quantities being on model full levels. Model full level l is located inside layer with interfaces $\tilde{l-1}$ and $\tilde{l}$.

Comment: There is only one integral operator $\underline{\mathbf{I_0^\eta}}$ defined in VFE scheme (RINTBF11). It represents integration from model top to any level η. The total integral over atmosphere $\underline{\mathbf{I_0^1}}$ is obtained when we evaluate $\underline{\mathbf{I_0^\eta}}$ on model surface ($\eta = 1$).

Conditions (3) and (4) express mass conservation

$$\underline{\mathbf{I_0^1}} \cdot \frac{\boldsymbol{\delta\pi}}{\boldsymbol{\delta\eta}} = \underline{\mathbf{I_0^1}} \cdot \frac{\boldsymbol{\delta A}}{\boldsymbol{\delta\eta}} + \pi_s \underline{\mathbf{I_0^1}} \cdot \frac{\boldsymbol{\delta B}}{\boldsymbol{\delta\eta}} = \pi_s, \tag{5}$$

and they are used to define full level differences δA_l and δB_l.

The depth of the layer $\delta\eta_l$is defined as

$$\delta\eta_l \quad = \quad \eta_{\tilde{l}} - \eta_{\tilde{l-1}}. \tag{6}$$

with half-level η defined as

$$\eta_{\tilde{l}} \quad = \quad \frac{A_{\tilde{l}}}{p_0} + B_{\tilde{l}} \tag{7}$$

with constant reference pressure $p_0 = 101325\,Pa$. Full level values of η required during construction of FE operators are

$$\eta_l \quad = \quad \frac{1}{2}\left(\eta_{\tilde{l-1}} + \eta_{\tilde{l}}\right). \tag{8}$$

The half-levels values $A_{\tilde{l}}$ and $B_{\tilde{l}}$ are known a priori. The first guess of differences is computed from them as

$$\hat{\delta A_l} \quad = \quad A_{\tilde{l}} - A_{\tilde{l-1}} \tag{9}$$

$$\hat{\delta B_l} \quad = \quad B_{\tilde{l}} - B_{\tilde{l-1}}. \tag{10}$$

In order to fullfill (4), the first guess obtained by (10) is integrated

$$\underline{\mathbf{I_0^1}} \cdot \frac{\boldsymbol{\delta\hat{B}}}{\boldsymbol{\delta\eta}} \quad = \quad \alpha, \tag{11}$$

and finally δB_l is determined

$$\delta B_l \quad = \quad \frac{\delta\hat{B}_l}{\alpha}, \tag{12}$$

and (4) is fullfilled exactly.

The ECMWF VFE implementation is based on geometrical properties of $\frac{\delta A}{\delta\eta}$ curve. When you see Figure 1), it is apparent that condition (3) is fullfiled when the area above the point where $\frac{\delta A}{\delta\eta} = 0$ is rescaled to be equal to area below that point. In [1], the red area is iteratively rescaled to be equal to

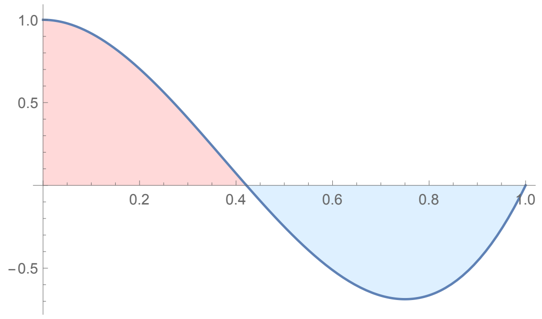


Figure 1: Typical shape of $\frac{\boldsymbol{\delta A}}{\boldsymbol{\delta \eta}}$. The condition $\underline{\mathbf{I}_{\mathbf{0}}^{\mathbf{1}}} \cdot \frac{\boldsymbol{\delta A}}{\boldsymbol{\delta \eta}} = 0$ represents condition that the red and blue area must be equal in their size.

blue one. We tested also another approaches (rescaling blue area resp. rescaling both by preserving total area), but it has no influence on result.

We use non iterative approach. We transform condition (3) into form

$$\underline{\mathbf{I}_{\mathbf{0}}^{\mathbf{1}}} \cdot \left(\frac{1}{p_0} \frac{\boldsymbol{\delta A}}{\boldsymbol{\delta \eta}} + 1 \right) = 1 \tag{13}$$

taking into account property $\underline{\mathbf{I}_{\mathbf{0}}^{\mathbf{1}}} 1 = 1$.

The differences δA_l are then computed from relation

$$\frac{1}{p_0} \frac{\delta A_l}{\delta \eta_l} + 1 = \frac{1}{\beta} \left(\frac{1}{p_0} \frac{\hat{\delta A_l}}{\delta \eta_l} + 1 \right) \tag{14}$$

with constant β computed from guess

$$\beta = \underline{\mathbf{I}_{\mathbf{0}}^{\mathbf{1}}} \cdot \left(\frac{1}{p_0} \frac{\hat{\boldsymbol{\delta A}}}{\boldsymbol{\delta \eta}} + 1 \right). \tag{15}$$

Condition (3) is then fullfilled exactly.

Comment: We could define condition for δA_l *as*

$$\underline{\mathbf{I}_{\mathbf{0}}^{\mathbf{1}}} \cdot \left(\frac{1}{p_0} \frac{\boldsymbol{\delta A}}{\boldsymbol{\delta \eta}} + \boldsymbol{g} \right) = 1 \tag{16}$$

with any auxiliary discrete function $\boldsymbol{g}$ *that satisfy condition* $\underline{\mathbf{I}_{\mathbf{0}}^{\mathbf{1}}} \boldsymbol{g} = 1$. *Natural choice could be* $\boldsymbol{g} = \frac{\boldsymbol{\delta B}}{\boldsymbol{\delta \eta}}$. *This was tested and there was no visible difference in results in Straka test when compared to choice* $\boldsymbol{g} = 1$.

Once we have correct values of full level depths we can compute full level values A_l and B_l as

$$\underline{\mathbf{I}_{\mathbf{0}}^{\eta}} \cdot \frac{\boldsymbol{\delta A}}{\boldsymbol{\delta \eta}} = A \tag{17}$$

$$\underline{\mathbf{I}_{\mathbf{0}}^{\eta}} \cdot \frac{\boldsymbol{\delta B}}{\boldsymbol{\delta \eta}} = B. \tag{18}$$

It allows us to compute full level hydrostatic pressure $\pi_l = A_l + B_l \pi_s$.

When using prognostic gw on half levels we need also the half-level $\frac{\tilde{\boldsymbol{\delta A}}}{\boldsymbol{\delta \eta}}$ and $\frac{\tilde{\boldsymbol{\delta B}}}{\boldsymbol{\delta \eta}}$ to evaluate $\boldsymbol{m_h}$ in vertical momentum prognostic equation

$$\frac{d\boldsymbol{w}}{dt} = \frac{g}{\boldsymbol{m_h}} \cdot \underline{\mathbf{D_h}} \cdot (\boldsymbol{p} - \boldsymbol{\pi}). \tag{19}$$

For details see [2]. This values are computed from spline fit of full level values used implicitly inside $\underline{\mathbf{I}_{\mathbf{0}}^{\eta}}$ operator. We design interpolation operator omitting mass and stiffness matrices as

$$\underline{\mathbf{T}} = \underline{\mathbf{A_h}} \cdot \underline{\mathbf{A}}^{-1}, \tag{20}$$

with $\{A\}_{kl} = a_k(\eta_l)$ and $\{A_h\}_{k\tilde{l}} = a_k(\eta_{\tilde{l}})$ being projections from full-level into FE space and from FE space back to half-levels otherwise.

The set of basis functions $a_k(\eta)$ and boundary conditions are the same as used for $\underline{\mathbf{I}_{\mathbf{0}}^{\eta}}$. The half level differences of A and B yields

$$\frac{\tilde{\boldsymbol{\delta A}}}{\boldsymbol{\delta \eta}} = \underline{\mathbf{T}} \frac{\boldsymbol{\delta A}}{\boldsymbol{\delta \eta}} \tag{21}$$

$$\frac{\tilde{\boldsymbol{\delta B}}}{\boldsymbol{\delta \eta}} = \underline{\mathbf{T}} \frac{\boldsymbol{\delta B}}{\boldsymbol{\delta \eta}}. \tag{22}$$

We tested this formulation in equation (19) with no visible influence on results. The name of transformation matrix $\mathbf{T}$ is RTRAFH. Routine SUVERTFEB has been modified to allow preparation of intepolation operators and modifications are done under key LVDA.

III. Set of operators with explicit input boundary conditions $f_s = f_L$ and $\frac{\partial f}{\partial \eta}_s = 0$

We have introduced the two boundary conditions at each material boundary (either model top or model bottom) into integral operator $\underline{\mathbf{I}_{\mathbf{0}}^{\eta}}$. We measure the quality of operator in term of smoothness of quantity $m = \frac{\partial \pi}{\partial \eta}$. This is important aspect to avoid numerical errors as this quantity appears in every term with any vertical operator. Taking into account that $m = \frac{\partial A}{\partial \eta} + \frac{\partial B}{\partial \eta} \pi_s$ and conditions (3) and (4), the smoothness of m is determined by the properties of $\underline{\mathbf{I}_{\mathbf{0}}^{\eta}}$ operator.

We studied the smoothness of spline approximation of m for A and B defined in analytical form

$$\omega(\eta) = \eta^2(3 - 2\eta) \tag{23}$$

$$A(\eta) = \pi_0 \eta (1 - \omega(\eta)) \tag{24}$$

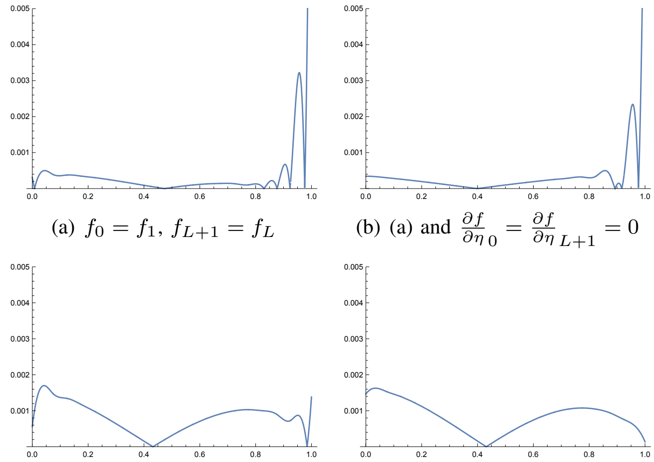
$$B(\eta) = \eta \omega(\eta), \tag{25}$$

with

$$m(\eta) = \pi_0 \left(8\eta^3 - 9\eta^2 + 1 \right) + \pi_s (9 - 8\eta) \eta^2. \tag{26}$$

When reference pressure π_0 is equal to surface pressure π_s the coordinate becomes pure σ coordinate with $m = \pi_s = const.$. We use $\pi_s = 90000\, Pa$ and $\pi_0 = 101325\, Pa$.

We sampled A and B on $L + 1$ half levels with regular distribution ($\eta_{\tilde{l}} = \frac{l}{L+1}$). Then we fit discrete values of $\frac{\delta A}{\delta \eta}$ and $\frac{\delta B}{\delta \eta}$ with spline with some set of explicit boundary conditions.



(a) $f_0 = f_1,\ f_{L+1} = f_L$

(b) (a) and $\frac{\partial f}{\partial \eta}_0 = \frac{\partial f}{\partial \eta}_{L+1} = 0$

(c) (a) and $\frac{\partial^2 f}{\partial \eta^2}_0 = \frac{\partial^2 f}{\partial \eta^2}_{L+1} = 0$

(d) (a) and $\frac{\partial^3 f}{\partial \eta^3}_0 = \frac{\partial^3 f}{\partial \eta^3}_{L+1} = 0$

Figure 2: Accuracy of spline fit of m with various boundary conditions.

If spline fit of function f is written as

$$S(f) = \sum_{i=0}^{L+1} \hat{f}_i e(\eta)_i. \qquad (27)$$

then we compute spline fit of m as

$$S(m) = S(\frac{\delta A}{\delta \eta}) + S(\frac{\delta B}{\delta \eta})\pi_s. \qquad (28)$$

This allows us to compute error of spline fit defined as $err(\eta) = |m(\eta) - S(m)|$, with $m(\eta)$ from (26). $err(\eta)$ for various boundary conditions is shown on Figure 2.

When we compute global error as integral of $err(\eta)$ over whole domain, we found that minimum absolute error is associated with boundary condition $\frac{\partial f}{\partial \eta}_0 = \frac{\partial f}{\partial \eta}_{L+1} = 0$ combined with $f_0 = f_1,\ f_{L+1} = f_L$.

This boundary conditions are used in set of operators RINTBF11, RDERBF11, RDDERBF11.

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