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Vertical finite elements discretization in NH model ALADIN
CY30T1

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1 Introduction

The vertical representation of the atmosphere in the NWP models must be accurate and efficient at the same time. The meso-scale convective phenomena requires higher vertical resolution on one side, but the number of vertical levels is limited by available computer performance and the complexity of microphysical parameterization. Therefore the high accuracy numerical scheme in vertical direction shall be applied.

The finite element (FE) method is well suited for the NWP. It can be easily implemented in irregular nodal space (traditional approach in NWP models) and on the other side it provides the two-times higher accuracy than the finite difference method on the same stencil. This property is known as a superconvergence.

The higher accuracy of vertical discretization ensures the quicker convergence of discretized system solution towards the analytical solution as $\delta\eta$ approaches 0. However, the aspect of stability must be considered for the systems with time evolution. When assuring the stability of the system with VFE we proceed in the same manner as it was done for the vertical finite difference (FD) model.

To analyze the stability of VFE discretization for various model time stepping is out of scope of this report. We have analyzed and studied the stability of two-time level non-extrapolating iterative centered implicit (2TL ICI) scheme. To consider VFE model stable it must be:

1. unconditionally stable with respect to stability of SI system with no explicitly or iteratively treated residual. In other words the eigenvalues of linear SI operator must be purely imaginary.
2. conditionally stable with respect to temperature SHB stability test with non-zero explicitly or iteratively treated residual. FD model is stable for $T_a^* < T < T^*$ for $n = 0, 2, 4, ...$ iterations and for $\frac{T_a^*}{2} < T < 2T^*$ for $n = 1, 3, 5, ...$ iterations of 2TL ICI scheme. We requires the similar properties for the model with VFE discretization and the same time stepping.
3. unconditionally stable with respect to pressure SHB instability in σ system. The stability of the 2TL ICI scheme must be independent on the choice of ${\pi_s}^*$ for σ coordinate.
4. neutral or weakly unstable with respect to pressure SHB stability test with non-zero explicitly or iteratively treated residual with η coordinate. The 2TL ICI scheme is always slightly unstable with respect to pressure SHB stability with η vertical coordinate. However, this instability is very weak for ${\pi_s}^* < \pi_s$ and controlled by other dissipative processes in ALADIN NH time stepping (SL advection, diffusion ...)

This report describes the implementation of the first version of VFE scheme into NH model ALADIN in CY30T1.

2 The choice of vertical coordinate

The atmosphere is divided into L layers and the model full levels ($l = 1, .., L$) are defined inside the layers and the half levels ($\tilde{l} = 0, .., L$) on layers interfaces.

The vertical coordinate η for VFE discretized model is implicitly defined on model full levels using the relation

$$\frac{\partial \pi}{\partial \eta}_l = \frac{\partial A}{\partial \eta}_l + \frac{\partial B}{\partial \eta}_l \pi_s.$$

The vertical derivatives $\frac{\partial A}{\partial \eta}_l$ and $\frac{\partial B}{\partial \eta}_l$ are derived from the prescribed half level values of $A_{\tilde{l}}$ and $B_{\tilde{l}}$. Vertical integrals of $\frac{\partial A}{\partial \eta}_l$ and $\frac{\partial B}{\partial \eta}_l$ computed with VFE integral operators must meet the conditions $\int_0^1 \frac{\partial A}{\partial \eta} d\eta = 0$ and $\int_0^1 \frac{\partial B}{\partial \eta} d\eta = 1$ in order to conserve the mass. The iterative procedure to compute $\frac{\partial A}{\partial \eta}_l$ and $\frac{\partial B}{\partial \eta}_l$ from $A_{\tilde{l}}$ and $B_{\tilde{l}}$ is coded in routine SUVERT and the detailed explanation is given in Untch and Hortal or in Benard.

The full level values of A_l and B_l are then computed with the VFE vertical integration of converged values of $\frac{\partial A}{\partial \eta}_l$ and $\frac{\partial B}{\partial \eta}_l$. The full level pressure is then

$$\pi = A_l + B_l \pi_s.$$

The prognostic variables and all diagnostic quantities are defined on the model full levels. The values of model half levels are not used in VFE discretization except the values at the top of the model ($\eta = 0$) and at the model surface ($\eta = 1$), if these are considered to be half levels.

The coordinate η is implicitly defined in previous expressions. The explicit values of η are required in SL scheme and also in VFE scheme. The computation is controlled by the key LREGETA, when true the vertical levels are regular otherwise they are derived using the values of pressure on model levels $\eta = \frac{A}{p_{00}} + B$, with p_{00} being auxiliary pressure.

3 Vertical finite element method

As already mentioned all the prognostic quantities in model ALADIN with VFE discretization are defined on model full levels. All derivatives and the integrals computed with VFE operators are valid on model full levels, except the integral evaluated over the whole model domain that is valid on model surface ($\eta = 1$).

We do evaluate only the vertical integrals and the vertical derivatives using the finite element (FE) technique. The products of variables are evaluated in the physical space.

3.1 VFE derivative operator

We want to compute the derivative $F = \frac{\partial f}{\partial \eta}$. The derivative F and the differentiated function f are continuous. We expand this functions in the series of basis functions with compact support (locally non-zero functions). We get

$$\sum_{i=1}^{N} \tilde{F}_i d_i = \sum_{i=1}^{N} \tilde{f}_i \frac{\partial e_i}{\partial \eta} + R.$$

The continuous F and f are generally not completely represented by the truncated series (only in the very special cases) and the truncation error R is not zero. The both are expanded using generally different set of basis functions $e_i, i = 1, N$ and $d_i, i = 1, N$.

In the discrete case the f and F are represented at full model levels $\eta_j, j = 1, N$. The transformation from the physical space to the FE space yields $\sum_{i=1}^{N} \tilde{f}_i e_i(\eta_j) = f(\eta_j)$ and $\sum_{i=1}^{N} \tilde{F}_i d_i(\eta_j) = F(\eta_j)$. Written in matrix form in gives

$$T_f \tilde{f} = f \tag{1}$$
$$T_F \tilde{F} = F \tag{2}$$

The finite element method is based on minimization of truncation error in the weighted integral sense

$$\int_0^1 R w_k \delta\eta =< R, w_k >= 0.$$

We integrated over the whole η domain from 0 at the top to 1 at the surface. The set of weighting functions $w_k, k = 1, N$ generally differs from the functional or derivative bases. Finally we write in the FE space

$$\sum_{i=1}^{N} \tilde{F}_i < d_i, w_k >= \sum_{i=1}^{N} \tilde{f}_i < \frac{\partial e_i}{\partial \eta}, w_k > \tag{3}$$

in matrix form

$$A\tilde{F} = B\tilde{f}. \tag{4}$$

Using (4) and (1) we can finally define the VFE derivative operators $\mathbf{D}$ as

$$F = T_F A^{-1} B T_f^{-1} f = \mathbf{D} f. \tag{5}$$

VFE derivative operator $\mathbf{D}$ returns the derivatives of input quantity on each model level. Operator acts on the vector of quantities defined on full levels.

3.2 VFE integral operator

The detailed description is given in Untch and Hortal or in Benard. The integral operator K returns the values of integral from the model surface to the model full levels and the model top as

$$F = T_F A^{-1} C T_f^{-1} f = \mathbf{F} f \tag{6}$$

with C being defined as

$$C =< \int_1^\eta e_i \delta\eta, w_k >$$

In further text we have adopted following three VFE integral operators.

- **F** - returns the value of integral from model top to the full level l. Operator acts on the vector of quantities defined on full levels.
- **J** - returns the value of integral from model surface to the model top. Operator acts on the vector of quantities defined on full levels. The operator **J** returns the vector that contains the same value for each level.
- **K** - returns the values of integral from model surface to the full level l. Operator acts on the vector of quantities defined on full levels. It is defined as $\mathbf{K} = \mathbf{J} - \mathbf{F}$.

3.3 The choice of basis functions

We have used the piecewise linear basis functions (NVDER=1) or the cubic B-splines basis functions (NVDER=3). Here we do concentrate on description of issues related to cubic B-splines choice.

We are using Galerkin method for which the functional basis, the derivative basis and the weighting functions are the same ($e_i = w_i = d_i$) for the computation of derivatives.

The situation for the integrals is different. The functional basis e_i is chosen such that the values of basis functions below the lowest full model level are constant. The weighting functions are defined zero outside the model domain $\eta > 1$ or $\eta < 0$.

3.4 The overdimensioning and the boundary conditions treatment

In order to cover the whole model domain $\eta \in < 0, 1 >$ with the complete functional base

- $L + 2$ basis functions is required for the linear base
- $L + 4$ basis functions is required for the cubic B-splines base.

In the case of cubic B-splines to perform transformations (1) we must know $L+4$ functional values, but we do have available only L values from model full levels. Therefore additional 4 functional values must be added.

In the case of VFE integral operator the vertical domain is extended above the model domain and below the model domain (see Untch and Hortal for additional information) and the following four assumptions is done

$$\begin{aligned} f(\eta_0) &= f(\eta_1) && (7) \\ f(\eta_{L+1}) &= f(\eta_L) && (8) \\ \frac{\partial f}{\partial \eta}_0 &= 0 && (9) \\ \frac{\partial f}{\partial \eta}_{L+1} &= 0. && (10) \end{aligned}$$

The assumptions done for the VFE derivative operator are different. The conditions are prescribed on the material boundaries (in our case at model top $\eta = 0$ and model surface $\eta = 1$). The additional 4 assumptions for the derivative calculations are

$$
\begin{aligned}
f(\eta_{\tilde{0}}) &= f(\eta_1) && (11)\\
f(\eta_{\tilde{L}}) &= f(\eta_L) && (12)\\
\frac{\partial f}{\partial \eta}_{\tilde{0}} &= 0 && (13)\\
\frac{\partial f}{\partial \eta}_{\tilde{L}} &= 0. && (14)
\end{aligned}
$$

This assumptions are not consistent for the time being with the calculation of vertical laplacian for example where the term $\frac{f^2}{m}\frac{\partial p-\pi}{\partial \eta}$ is not zero at the model surface. The choice of these additional assumption and their influence on the model properties must be further studied.

4 Linear operator discretization

The model is formulated using prognostic variables d_4 and $\hat{q}$ (NVDVAR=4 and NPDVAR=2). The linear operator formulated with this two NH prognostic variables and temperature T, divergence D and $q_s = ln(\pi_s)$ yields

$$
\begin{aligned}
\frac{\partial D}{\partial t} &= -R\mathbf{G}^* \triangle T + RT^*(\mathbf{G}^* - 1)\triangle \hat{q} - RT^* \triangle q_s - \triangle \phi_s \\
\frac{\partial T}{\partial t} &= -\frac{RT^*}{C_v}\left(\mathbf{m}_*{}^2 D + d_4\right) \\
\frac{\partial q_s}{\partial t} &= -\mathbf{N}^* \mathbf{m}_*{}^2 D \\
\frac{\partial d_4}{\partial t} &= -\frac{g^2}{RT_a{}^*}\mathbf{L}^* \hat{q} \\
\frac{\partial \hat{q}}{\partial t} &= -\frac{C_p}{C_v}(\mathbf{m}_*{}^2 D + d_4) + \mathbf{S}^* \mathbf{m}_*{}^2 D.
\end{aligned}
\qquad (15)
$$

The vertical linear operators on level l are defined

$$
\begin{aligned}
(\mathbf{G}^*\psi)_l &= \mathbf{K}\left(\frac{1}{\pi^*}\frac{\partial \pi^*}{\partial \eta}\psi\right)_l && (16)\\
(\mathbf{S}^*\psi)_l &= \frac{1}{\pi^*_l}\mathbf{F}\left(\frac{\partial \pi^*}{\partial \eta}\psi\right)_l && (17)\\
(\mathbf{N}^*\psi)_l &= \frac{1}{\pi_s{}^*}\mathbf{J}\left(\frac{\partial \pi^*}{\partial \eta}\psi\right)_l && (18)\\
(\mathbf{L}^*\psi)_l &= \mathbf{D}\left[\delta\eta\frac{g^2}{\delta\pi^*}\mathbf{D}\left(\pi^*\psi\right)\right]_l && (19)
\end{aligned}
$$

The elimination of linear operator is performed to acquire the equations for the vertical divergence d_3 and the horizontal divergence D (details see Benard internal documentation). This gives

$$
\begin{aligned}
\left[I - \tau^2\left(c^{*2} - RT^*C1\right)\triangle \mathbf{m}_*{}^2\right] D &= D^\bullet + \tau^2 \triangle\left[\left(-RT^*\mathbf{G}^* + c^{*2}\right)\right] d_3 \\
\left(1 - \tau^2\frac{c^{*2}}{H^*H_a{}^*}\mathbf{L}^*\right) d_3 &= d_3{}^\bullet + \tau^2\frac{1}{H_a{}^*}\mathbf{L}^*\frac{1}{H^*}\left(-RT^*\mathbf{S}^* + c^{*2}\right)\mathbf{m}_*{}^2 D,
\end{aligned}
\qquad (20)
$$

with C1 being the constrain that is in the continuous case equal to zero

$$C1 : (\mathbf{G}^*\mathbf{S}^* - \mathbf{G}^* - \mathbf{S}^* + \mathbf{N}^*)\psi = 0. \tag{21}$$

The FD discretization of NH ALADIN has been designed to keep C1 zero, and in such case the system of the two equations can be further eliminated to the single structure equation for variable d_3.

So far, we did not succeed to find such definition of VFE integral operators that would satisfy the constrain C1. Therefore the further elimination of the system is not feasible and the system of two equations must be solve in the spectral space.

The spectral operator is stored in array SIEHEL and it is defined in the routine SUEHEL.

5 Non-linear model discretization

The system of nonlinear prognostic equations formulated with the prognostic variables $u, v, T, q_s, d_4, \hat{q}$ is

$$\begin{aligned}
\frac{du}{dt} &= -\frac{RT}{p}\frac{\partial p}{\partial x} - \left(1 + \frac{1}{m}\frac{\partial(p-\pi)}{\partial\eta}\right)\frac{\partial\phi}{\partial x} + KC_u(\lambda,\varphi) + fv + \left(\frac{\partial u}{\partial t}\right)^{h.diff} \\
\frac{dv}{dt} &= -\frac{RT}{p}\frac{\partial p}{\partial y} - \left(1 + \frac{1}{m}\frac{\partial(p-\pi)}{\partial\eta}\right)\frac{\partial\phi}{\partial y} + KC_v(\lambda,\varphi) - fu + \left(\frac{\partial v}{\partial t}\right)^{h.diff} \\
\frac{dT}{dt} &= -\frac{R}{C_vT}D_3 \\
\frac{\partial q_s}{\partial t} &= -\frac{1}{\pi_s}\int_0^1 \vec{\nabla}\,(m\vec{v})\,d\eta \\
\frac{dd_4}{dt} &= -\frac{g^2p}{mRT}\frac{\partial}{\partial\eta}\left(\frac{1}{m}\frac{\partial(p-\pi)}{\partial\eta}\right) + \frac{gp}{mRT}\vec{\nabla}w\frac{\partial\vec{v}}{\partial\eta} - d_3D_3 + d_3D + \frac{d\mathcal{X}}{dt} \\
\frac{d\hat{q}}{dt} &= \frac{C_p}{C_v}D_3 - \frac{\omega}{\pi}
\end{aligned} \tag{22}$$

it is completed by the following diagnostic relations

$$\begin{aligned}
\pi_s &= e^{q_s} \\
\pi &= A + B\pi_s \\
p &= \pi e^{\hat{q}} \\
\phi &= \phi_s + \int_\eta^1 \frac{mRT}{p}d\eta \\
\mathcal{X} &= \frac{p}{mRT}\vec{\nabla}\phi\frac{\partial\vec{v}}{\partial\eta} \\
D_3 &= D + d_4 \\
w &= \vec{v}_s\vec{\nabla}\phi_s + \frac{1}{g}\int_\eta^1 \frac{mRT}{p}(d_4 - \mathcal{X})d\eta \\
\frac{\omega}{\pi} &= \frac{1}{\pi}\left[\vec{v}\vec{\nabla}\pi - \int_0^\eta \vec{\nabla}(m\vec{v})\,d\eta\right] \\
m\dot{\eta} &= B\int_0^1 \vec{\nabla}\,(m\vec{v})\,d\eta - \int_0^\eta \vec{\nabla}\,(m\vec{v})\,d\eta.
\end{aligned} \tag{23}$$

The red terms are affected by the change of vertical discretization from FD to VFE scheme.

Full level geopotential computation

$$\phi_l = \phi_s + \mathbf{K}\left(\frac{mRT}{p}\right)_l$$

The horizontal gradient of geopotential on full levels is

$$\vec{\nabla}\phi_l = \vec{\nabla}\phi_s + \mathbf{K}\left[\frac{mRT}{p}\left(\frac{\vec{\nabla}m}{m} + \frac{\vec{\nabla}RT}{RT} - \frac{\vec{\nabla}p}{p}\right)\right]_l$$

Usage the prognostic variable $\hat{q} = ln\left(\frac{p}{\pi}\right)$ and definition of vertical variable η gives

$$\vec{\nabla}\phi_l = \vec{\nabla}\phi_s + \mathbf{K}\left[\frac{mRT}{p}\left(\frac{1}{m}\frac{\partial B}{\partial \eta}\vec{\nabla}\pi_s + \frac{\vec{\nabla}RT}{RT} - B\frac{\vec{\nabla}\pi_s}{\pi} - \vec{\nabla}\hat{q}\right)\right]_l$$

The previous expression is coded in the routine GPRGEO. The exactly following expression is coded in model version cy30T1

$$\vec{\nabla}\phi_j = \vec{\nabla}\phi_s + \mathbf{K}\frac{1}{\delta\eta_l}\left[-\frac{\delta\pi_l}{\pi_l}\frac{\pi_l}{p_l}\left(\vec{\nabla}RT - RT_l\vec{\nabla}\hat{q}\right) - \frac{\pi_l}{p_l}RT_l\frac{\delta\pi_l}{\pi_l}\left(\frac{\delta B_l}{\delta\pi_l} - \frac{B_l}{\pi_l}\right)\vec{\nabla}\pi_s\right]_j$$

The definition of used variables in routine GPRGEO is in the following table

Variable	Content
PLNPR	$\frac{\delta\pi_l}{\pi_l}$
PREHYDSPRE	$\frac{\pi_l}{p_l}$
(PLNPRL,PRNPLM)	$\frac{\delta\pi_l}{\pi_l}\left(\frac{\delta B_l}{\delta\pi_l} - \frac{B_l}{\pi_l}\right)\vec{\nabla}\pi_s$
VRDETAH	$\frac{1}{\delta\eta_l}$
(PQCHAL,PQCHAM)	$\vec{\nabla}\hat{q}$
(PRTL,PRTM)	$\vec{\nabla}RT$

The analogy with the existing code would require to define the mass weighted vertical integral operator $\mathbf{G}$ as $\mathbf{G}\psi = \mathbf{K}\,(m\psi)$.

The pressure gradient force then gives

$$PGF_l = RT_l\left(\frac{B_l}{\pi_l}\vec{\nabla}\pi_s + \vec{\nabla}\hat{q}_l\right) + \frac{p_l}{\pi_l}\left(1 + \pi_l\mathbf{D}(\hat{q})_l\right)\vec{\nabla}\phi_l$$

The three-dimensional divergence in mass-based η coordinate discretized in VFE manner is

$$D3_l = D_l + \delta\eta_l\frac{p_l}{\pi_l}\frac{\pi_l}{\delta\pi_l RT_l}\vec{\nabla}\phi_l\mathbf{D}(\vec{v})_l + d_{3l} = D_l + d_{4l}$$

The horizontal gradient of true vertical velocity is obtained by integration of prognostic variable d_4 as

$$\vec{\nabla} gw_l = \vec{\nabla} gw_s + \mathbf{K}\left[\frac{1}{\delta\eta}\frac{\pi}{p}\frac{\delta\pi RT}{\pi}\left(d_4 - \mathcal{X}\right)\left(\left(\frac{\delta B_l}{\delta\pi_l} - \frac{B_l}{\pi_l}\right)\vec{\nabla}\pi_s + \frac{\vec{\nabla} RT}{RT} + \frac{\vec{\nabla} d_4 - \vec{\nabla}\mathcal{X}}{d_4 - \mathcal{X}} - \vec{\nabla}\hat{q}\right)\right]_l.$$

So called Z-term in the vertical momentum equation is discretized as

$$Z_l = \delta\eta_l \frac{p_l}{\pi_l}\frac{\pi_l}{\delta\pi_l RT_l}\vec{\nabla} gw_l \mathbf{D}(\vec{v})_l$$

The vertical velocity multiplied by gravity constant is the half level quantity in the model with Lorentz staggering. The adiabatic tendency is therefore computed on the model half levels as well. Because the surface is the model lowest half level the incorporation of kinematic BBC is straightforward in that case.

However, in VFE discretization all quantities are defined on the model full levels. The full level derivatives are computed from the full level functional values. It is not apparent how to include kinematic BBC into VFE scheme.

5.1 Vertical laplacian - version 1

To overcome the problem we could accept the mixed solution. Firstly we compute the half level adiabatic tendencies of gw with the kinematic BBC at the model surface in the same manner as it is done with finite differences. Then we compute the vertical derivative of half level adiabatic tendencies using the finite difference approach

$$\frac{\partial \dot{gw}}{\partial\eta}_l = \frac{\dot{gw}_{\tilde{l}} - \dot{gw}_{\tilde{l}-1}}{\eta_{\tilde{l}} - \eta_{\tilde{l}-1}}.$$

The $\frac{\partial \dot{gw}}{\partial\eta}_l$ is full level quantity and can be integrated using VFE integral operator **K** from model surface to the full levels. This gives the adiabatic tendency of gw at model full level l

$$\dot{gw}_l = \dot{gw}_s + \mathbf{K}\left(\frac{\partial \dot{gw}}{\partial\eta}\right)_l.$$

The kinematic BBC is used to express surface term $\dot{gw}_s$. The same formulation as in the case of FD discretization is used.

Finally the non-linear vertical laplacian on full levels is computed as

$$\mathbf{L}_l = \mathbf{D}\left(\dot{gw}\right)_l$$

Accuracy and convergence of solution as number of levels goes to infinity (TODO).

5.2 Vertical laplacian - version 2

The second approach is more in the VFE spirit, but only the first derivative VFE operator **D** is used. The discretization is

$$\mathbf{L}_l = \mathbf{D}\left[\delta\eta\frac{g^2}{\delta\pi}\mathbf{D}\left(p - \pi\right)\right]_l$$

The stability of SI scheme requires that the eigenvalues of the linear laplacian operator are purely real and negative. However, the laplacian discretized in the previous manner violates this condition.

The real negative eigenvalues spectra is obtain
The adiabatic tendency of d_3 variable is then derived using purely VFE approach as

$$\dot{d}_{3l} = \mathbf{L}_l + Z_l + {d_3}^2.$$

Here the Z_l is the so-called Z-term. It is present at the RHS in any system formulated with the vertical divergence as a prognostic variable. It is discretized in VFE manner as

$$Z_l = \delta\eta_l \frac{p_l}{\pi_l} \frac{\pi_l}{\delta\pi_l RT_l} \vec{\nabla} gw_l \mathbf{D}(\vec{v})_l$$

Using the previous approach we loose the theoretical eight order accuracy of VFE scheme based in cubic B-splines (this accuracy is reached only for regular vertical levels distribution).

The crucial for the moment accuracy is the accuracy of vertical laplacian discretization as it is the only vertical operator present in the vertical structure equation. The laplacian is implicit in the previous equation and its explicit form is

$$\overline{\mathbf{L}}\psi_l = \mathbf{D}\left(g\dot{w}_s + \mathbf{K}\left(\frac{\partial g\dot{w}}{\partial \eta}\right)\right)_l.$$

The accuracy of laplacian can assessed in the σ coordinate where the analytical solution for prescribed ψ can be computed. On the following figure the comparison of accuracy of (TODO).

The same approach is used for the semi-implicit linear operator. It is derived above the flat surface ${\phi_s}^* = const.$ and the kinematic BBC is equal to 0.

6 Implementation in cy30T1

The VFE scheme is activated by the key LVERTFE in NAMCT0 namelist. Necessary choices in the case of VFE scheme for NH dynamics

- the cubic basis function must be used (NVSCH=3 in suct0)
- the simple computation of full level pressure must be on (LAPRXPK=TRUE in suct0)
- in this case the value of NDLNPR is ignored as it is consider to be relevant for the finite difference schemes
- the system of two vertically dependent equation is solved in the spectral space (L2LSOLVER=TRUE in namdyna) as the constrain C1 is not fulfilled in order to eliminate for the single variable structure equation
- VFE scheme works for the EUL and SL2TL schemes

7 Tests - validation

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