Two-time level iterative centered implicit semi-Lagrangian scheme with grid-point point prognostic variable gw and semi-implicit variable d_3 or d_4.

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Prague,2002 and Toulouse,2005

1 Motivation

When the idealized flow over Agnesi shape mountain was computed with the older version of NH model ALADIN using SL scheme, then the unrealistic "chimney" appeared directly over the mountain. The formulation of kinematic bottom boundary condition was expected to be the source of problems. Smith and Brozkova proposed to use the presented gw scheme to overcome this problem. The main motivation has been probably the fact that the bottom boundary condition in this case is significantly simplified.

The problem of chimneys in SL schemes without horizontal diffusion has been later solved with the new BBC condition (LRDBBC) (Smolikova and Brozkova). At that stage it seemed that gw scheme is not anymore needed. But the warm and cold bubble case has been studied by Vivoda and the erroneous behavior of this experiment with traditional schemes was reported. On other side gw scheme gives the correct results. This behavior has not been explained yet and it is still subject to be studied.

The gw scheme can be considered as a special case of d_4 implementation (hypothetic ND4SYS=5). So far the most effective one if we consider two-time level non-extrapolating gw scheme (2TL ICI NESC gw scheme).

There are the following crucial advantages of gw scheme comparing to the traditional scheme

- no need for auxiliary prognostic variable (AUX) for SL scheme with d_4 prognostic variable. The AUX is introduced to compute $\vec{\nabla} w$, that is not present in gw formulation. However, with Eulerian scheme auxiliary variable is needed also for gw to compute Eulerian advection term.
- no need for complex BBC condition (LRDBBC) to remove SL "chimneys"

- gw in the grid point space allows the physical tendency of vertical velocity to be computed directly by physical parameterizations, instead of d variable tendency
- smaller truncation error of space discretization, because the equations are simpler
- 2TL ICI NESC gw scheme is cheaper that the traditional one in terms of CPU

There are the following disadvantages of gw scheme comparing to the traditional scheme

- additional interpolations on half levels. The half level trajectory is computed as average between the trajectories from the adjacent vertical full levels (Smith and Brozkova). The whole set of half level interpolation weights is introduced.
- need to developed the new NXLAG option for the for extrapolating or leapfrog schemes

2 Notation

The notation $\mathcal{X}$ is used for quantity

$$\mathcal{X} = \frac{p}{mR_dT}\vec{\nabla}\phi\frac{\partial\vec{v}}{\partial\eta},$$

which appears in the elastic term. This quantity plays important role in the model ALADIN. It affects the model stability and its implicit treatment is assured via prognostic variable d_4.

The name d variable is used for prognostic variable related to vertical divergence ($\frac{\partial gw}{\partial\eta}$). It represents one of the variables $\hat{d}, d_3, d_4$. At the places which are specific for each variable the exact variable is mentioned. The variables are defined as following

$$\hat{d} = -\frac{\pi^*}{m^*R_dT^*}\frac{\partial gw}{\partial\eta} \quad (1)$$

$$d_3 = -\frac{p}{mR_dT}\frac{\partial gw}{\partial\eta} \quad (2)$$

$$d_4 = -\frac{p}{mR_dT}\frac{\partial gw}{\partial\eta} + \mathcal{X} \quad (3)$$

The notation X is used for the model state defined with the prognostic variable d,

$$X = (U, V, T, ln(\pi_s), \hat{q}, d).$$

The state vector Y is the model state defined with the prognostic variable gw,

$$Y = (U, V, T, ln(\pi_s), \hat{q}, gw).$$

The transformation between this two states is denoted as $X = f(Y)$. The transformation is defined in (1)-(3) for each d prognostic variable. This notation is used to emphasize that the all prognostic variables are involved in the transformation between the d variable and the gw variable

$$X = (U, V, T, ln(\pi_s), \hat{q}, f(U, V, T, ln(\pi_s), \hat{q}, gw)). \tag{4}$$

The notation $\mathcal{M}_d$ resp. $\mathcal{M}_w$ is used for the nonlinear model formulated with the X resp. Y. It describes the time evolution of states X resp. Y as

$$\tfrac{dX}{dt} = \mathcal{M}_d(X) \quad \text{resp.} \quad \tfrac{dY}{dt} = \mathcal{M}_w(Y).$$

The notation $\mathcal{B}_d$ resp. $\mathcal{B}_w$ denotes the linear operator for the model defined with X resp. Y. The operator $\mathcal{B}_d$ determines the model spectral kernel around which the iterations are performed. The relation $\mathcal{B}_d = f(\mathcal{B}_w)$ is valid only for the $\hat{d}$ prognostic variable.

3 Formulation

The main idea of gw scheme is that the part of the grid-point calculations are performed with the vertical velocity as a prognostic variable. Before going to spectral space the quantities formulated for w variable are transformed into quantities valid for d prognostic variable ($\hat{d}$, d_3 or d_4). The spectral space computations are then performed with d as a prognostic variable. The scheme is called gw scheme, because we use as a prognostic variable vertical velocity multiplied by the gravity constant.

The traditional 2TL ICI SL scheme (formulated with d scheme) can be written as (n denotes the current iteration ($n = 0$ for predictor))

$$\begin{aligned} P: \quad & (I - \tfrac{\Delta t}{2}\mathcal{B}_d)X_F^{t+\Delta t(0)} = X_O^t + \Delta t \mathcal{M}_d(X)_M^{t+\frac{\Delta t}{2}} + \tfrac{\Delta t}{2}\mathcal{B}_d\left(X_O^t - 2X_M^{t+\frac{\Delta t}{2}}\right) \\ C(n): \quad & (I - \tfrac{\Delta t}{2}\mathcal{B}_d)X_F^{t+\Delta t(n)} = X_O^t + \tfrac{\Delta t}{2}\left(\mathcal{M}_d(X)_F^{t+\Delta t(n-1)} + \mathcal{M}_d(X)_O^t\right) - \tfrac{\Delta t}{2}\mathcal{B}_d X_F^{t+\Delta t(n-1)} \end{aligned} \tag{5}$$

This scheme has been proved to be stable and consistent and therefore convergent (in the sense that it converges to the continuous equations solution).

The formulation (5) is independent on the computation of terms $\mathcal{M}_d(X)_M^{t+\frac{\Delta t}{2}}$ and $X_M^{t+\frac{\Delta t}{2}}$, and the statements valid in this section are valid for extrapolating and non-extrapolating schemes as well.

Stability:
With the current assumption made in the model ALADIN, a sufficient stability can be achieved only if the spectral calculation are performed with d_3 resp. d_4 as prognostic variable. It is not possible at this stage to formulate stable NH model ALADIN with the spectral solver formulated with gw prognostic variable. Such solution would be equivalent to the solution with $\hat{d}$ variable which was found to be stable only for small CFL and it is not satisfactory solution.

Consequence of abovementioned facts is that in the traditional and also in the *gw* scheme we must solve the following spectral space system

$$(I - \frac{\Delta t}{2}\mathcal{B}_d)X_F^{t+\Delta t(n)} = RHS_d \tag{6}$$

Consistency:
The linear part of the scheme must satisfy

$$\begin{array}{ll} P: & \lim_{\Delta t \to 0} \mathcal{B}_d \left(\frac{X^{t+\Delta t(0)} + X^t}{2} - X^{t+\frac{\Delta t}{2}} \right) = 0 \\ C(n): & \lim_{\Delta t \to 0} \mathcal{B}_d \left(\frac{X^{t+\Delta t(n)} - X^{t+\Delta t(n-1)}}{2} \right) = 0 \end{array} \tag{7}$$

in order to converge to the continuous system of equations

$$\frac{dX}{dt} = \mathcal{M}_d(X).$$

Two versions of the *gw* scheme have been implemented into the model ALADIN in the period 2001-2005. The first version has been coded by Smith with 3TL SL SI time stepping. As the model ALADIN evolved it was found that this version is not appropriate for the practical exploitation. Therefore Vivoda has coded the second version into CY29T2. It was *gw* scheme with the 2TL non-extrapolating (NESC) ICI SL time stepping.

The both versions are reported here for completeness and better understanding. The both approaches are applied on 2TL ICI time stepping.

3.1 Version 1: Transformation of RHS

This was the first solution implemented into ALADIN by Smith (around 2001). The idea was to compute the complete RHS with *gw* prognostic variable as

$$\begin{array}{ll} P: & RHS_w = Y_O^t + \Delta t \mathcal{M}_w(Y)_M^{t+\frac{\Delta t}{2}} + \frac{\Delta t}{2}\mathcal{B}_w \left(Y_O^t - 2Y_M^{t+\frac{\Delta t}{2}} \right) \\ C(n): & RHS_w = Y_O^t + \frac{\Delta t}{2} \left(\mathcal{M}_w(Y)_F^{t+\Delta t(n-1)} + \mathcal{M}_w(Y)_O^t \right) - \frac{\Delta t}{2}\mathcal{B}_w \left(Y_F^{t+\Delta t(n-1)} \right). \end{array}$$

It was further assumed that the transformation $X = f(Y)$ is valid for the RHS as well

$$RHS_d = f(RHS_w).$$

With this approach the consistency condition (7) can be satisfied only if

$$\mathcal{B}_d X = f(\mathcal{B}_w Y). \tag{8}$$

This condition is satisfied only for the $\hat{d}$ prognostic variable as was pointed by Smith (personal communication). For system with $\hat{d}$, thanks to (8), we then obtain

$$
\begin{aligned}
P: \quad & (I - \tfrac{\Delta t}{2}\mathcal{B}_d)X_F^{t+\Delta t(0)} = f_{\hat{d}}(Y_O^t + \Delta t \mathcal{M}_w(Y)_M^{t+\frac{\Delta t}{2}}) + \tfrac{\Delta t}{2}\mathcal{B}_d\left(X_O^t - 2X_M^{t+\frac{\Delta t}{2}}\right) \\
C(n): \quad & (I - \tfrac{\Delta t}{2}\mathcal{B}_d)X_F^{t+\Delta t(n)} = f_{\hat{d}}\left(Y_O^t + \tfrac{\Delta t}{2}\mathcal{M}_w(Y)_F^{t+\Delta t(n-1)} + \tfrac{\Delta t}{2}\mathcal{M}_w(Y)_O^t\right) - \tfrac{\Delta t}{2}\mathcal{B}_d X_F^{t+\Delta t(n-1)}
\end{aligned} \tag{9}
$$

This approach cannot be used for d_3 and d_4 since the schemes would not be consistent as

$$
\begin{aligned}
P: \quad & \lim_{\Delta t \to 0}\left(\tfrac{\mathcal{B}_d X^{t+\Delta t(0)} + f(\mathcal{B}_w X^t)}{2} - f(\mathcal{B}_w X^{t+\frac{\Delta t}{2}})\right) \neq 0 \\
C(n): \quad & \lim_{\Delta t \to 0}\left(\tfrac{\mathcal{B}_d X^{t+\Delta t(n)} - f(\mathcal{B}_w X^{t+\Delta t(n-1)})}{2}\right) \neq 0
\end{aligned} \tag{10}
$$

3.2 Version 2: Transformation of explicit guesses

Looking at (9), it is clear that the linear terms $\mathcal{B}_d$ remains unaffected by the gw scheme in the case of $\hat{d}$ prognostic variable. It is necessary condition to satisfy stability and consistency conditions (6) and (7).

The new version of gw scheme is therefore as follows

$$
\begin{aligned}
P: \quad & (I - \tfrac{\Delta t}{2}\mathcal{B}_d)X_F^{t+\Delta t(0)} = f(Y_O^t + \Delta t \mathcal{M}_w(Y)_M^{t+\frac{\Delta t}{2}}) + \tfrac{\Delta t}{2}\mathcal{B}_d\left(X_O^t - 2X_M^{t+\frac{\Delta t}{2}}\right) \\
C(n): \quad & (I - \tfrac{\Delta t}{2}\mathcal{B}_d)X_F^{t+\Delta t(n)} = f(Y_O^t + \tfrac{\Delta t}{2}\mathcal{M}_w(Y)_F^{t+\Delta t(n-1)} + \tfrac{\Delta t}{2}\mathcal{M}_w(Y)_O^t) - \tfrac{\Delta t}{2}\mathcal{B}_d X_F^{t+\Delta t(n-1)}
\end{aligned} \tag{11}
$$

Such scheme is supposed to be as stable and consistent as the traditional scheme itself. As a first step, all the prognostic variables are evolved in time using the explicit time step

$$
\begin{aligned}
P: \quad & \tilde{Y}_F^{t+\Delta t(0)} = Y_O^t + \Delta t \mathcal{M}_w(Y)_M^{t+\frac{\Delta t}{2}} \\
C(n): \quad & \tilde{Y}_F^{t+\Delta t(n)} = Y_O^t + \tfrac{\Delta t}{2}\mathcal{M}_w(Y)_F^{t+\Delta t(n-1)} + \tfrac{\Delta t}{2}\mathcal{M}_w(Y)_O^t
\end{aligned}
$$

then the transformation is performed using explicit estimates

$$
\tilde{X}_F^{t+\Delta t} = f(\tilde{Y}_F^{t+\Delta t}). \tag{12}
$$

It should be outlined that the code has been modified for every prognostic equation, because it was thought that no linear term should enter in the explicit guess. Maybe other possibilities could be considered.

The linear terms are evaluated independently directly with X and then the RHS_d is computed as

$$
\begin{aligned}
P: \quad & RHS_d = \tilde{X}_F^{t+\Delta t} + \tfrac{\Delta t}{2}\mathcal{B}_d\left(X_O^t - 2X_M^{t+\frac{\Delta t}{2}}\right) \\
C(n): \quad & RHS_d = \tilde{X}_F^{t+\Delta t} - \tfrac{\Delta t}{2}\mathcal{B}_d X_F^{t+\Delta t(n-1)}
\end{aligned}
$$

Specificities for d_4 pognostic variable

The transformation (12) contains the explicit guess of $\mathcal{X}$ in the case where the prognostic variable d_4 is used. Then the transformation (3) gives

$$\tilde{d}_{4F}^{\,t+\Delta t} = \tilde{d}_{3F}^{\,t+\Delta t} + \tilde{\mathcal{X}}_F^{t+\Delta t}.$$

But the term $\tilde{\mathcal{X}}_F^{t+\Delta t}$ contains horizontal derivatives of the explicit estimate $\tilde{Y}^{t+\Delta t}$. They are not available in the grid point space where the transformation is performed. Therefore an approximative solution has to be adopted

$$\begin{aligned} P: &\qquad \tilde{\mathcal{X}}_F^{t+\Delta t} = \mathcal{X}_F^t \\ C(n): &\qquad \tilde{\mathcal{X}}_F^{t+\Delta t} = \mathcal{X}_F^{t+\Delta t(n-1)}. \end{aligned} \tag{13}$$

The predictor is treated in a non-extrapolating manner.

Problem of additional interpolations during predictor

Separation of the linear terms from the non-linear ones leads to complications for the SL scheme (personal communication with Smolikova and Brozkova). This is particular problem for the schemes where the non-linear terms and also the linear ones have to be both interpolated at origin point of SL trajectories. These terms cannot be grouped and interpolated together, but they have to be interpolated separately. This problem exists in the predictor step of 2TL SL scheme with Robert or LSETTLS extrapolation and in the predictor step of 3TL SL scheme as well.

The problem of additional interpolations can be easily overcome by more appropriate redistribution of terms into SL buffers than it is done for $NXLAG = 3$ in the current code. The explicit guess could be interpolated using high order interpolations and the linear terms could be interpolated using the linear interpolations. With such approach no additional interpolations or buffers are required.

Karim's comment:
NXLAG must remain independent of LPC_FULL, LPC_NESC or LGWADV. The new distribution of interpolations must remain under NXLAG=3 and must be controlled by LGWADV and NCURRENT_ITER.

Above mentioned problem doesn‘t exist in the predictor of 2TL NESC scheme , because in this case all the linear terms are evaluated at the final point of SL trajectories

$$\frac{\Delta t}{2}\mathcal{B}_d\left(X_O^t - 2X_M^{t+\frac{\Delta t}{2}}\right) = -\frac{\Delta t}{2}\mathcal{B}_d X_F^t,$$

During the corrector all linear terms are evaluated at the final point for all 2TL SL schemes and 3TL SL schemes and the problem of additional interpolations doesn't exist.

$$\frac{\Delta t}{2}\mathcal{B}_d\left(X_O^t - 2X_M^{t+\frac{\Delta t}{2}}\right) = -\frac{\Delta t}{2}\mathcal{B}_d X_F^{t+\Delta t(n-1)},$$

Only the 2TL NESC ICI scheme has been implemented into CY29T2 due to previously mentioned advantage.

4 Vertical discretization aspects

d prognostic variable is located at the full levels of Lorenz vertical grid used in NH model ALADIN, while the prognostic quantity gw is located at the half levels. This brings the additional problem into the SL advection treatment. The set of new trajectories have to be computed for the half levels, together with the interpolation points. Smith and Brozkova studied this problem and they found out that it is sufficient to compute the half level trajectories as a average between trajectories from the adjacent full levels

$$\vec{OF}_{\tilde{l}} = \frac{\vec{OF}_l + \vec{OF}_{l+1}}{2} \tag{14}$$

with the top boundary condition $\vec{OF}_{\tilde{0}} = \vec{OF}_1$ and the bottom boundary condition $\vec{OF}_{\tilde{L}} = \vec{OF}_L$.

In $\mathcal{M}_w$ is the prognostic equation for d variable replaced by the prognostic equation for gw

$$\frac{dgw_{\tilde{l}}}{dt} = \frac{g^2}{m}\frac{\partial(p-\pi)}{\partial\eta}.$$

The prognostic equation for gw is vertically discretized on half level $\tilde{l}$ as

$$\frac{dgw_{\tilde{l}}}{dt} = g^2\frac{(p-\pi)_{l+1} - (p-\pi)_l}{\pi_{l+1} - \pi_l}. \tag{15}$$

At the bottom of the atmosphere we assume that the surface horizontal wind $\vec{v}_{\tilde{L}}$ is equal to the wind at the first full layer above the surface $\vec{v}_L$. The free-slip BBC then gives

$$gw_{\tilde{L}} = \vec{v}_L\vec{\nabla}(\phi_{\tilde{L}}).$$

The bottom boundary condition for (15) is derived from free-slip BBC as

$$\frac{dgw_{\tilde{L}}}{dt} = \frac{d\vec{v}_L}{dt}\vec{\nabla}(\phi_{\tilde{L}}) + u^2\frac{\partial^2\phi_{\tilde{L}}}{\partial x^2} + 2uv\frac{\partial^2\phi_{\tilde{L}}}{\partial xy} + v^2\frac{\partial^2\phi_{\tilde{L}}}{\partial y^2}.$$

The top boundary condition is the Dirichlet one with the zero pressure (hydrostatic and non-hydrostatic as well) at the top of the model. This gives

$$\frac{dgw_{\tilde{0}}}{dt} = g^2\frac{(p-\pi)_1}{\pi_1}. \tag{16}$$

5 Implementation and data flow in CY29T2

Only the 2TL ICI NESC *gw* scheme has been implemented with d_3 or d_4 into CY29T2.

At informatics level the 2TL ICI NESC scheme is turned on with the following choices in the model namelist

NAMDYNA	
LGWADV	TRUE
NVDVAR	3 or 4
LRDBBC	FALSE
NAMCT0	
LSLAG	TRUE
LTWOTL	TRUE
LPC_FULL	TRUE
LPC_NESC	TRUE
NAMDYN	
XIDT	0.0
VESL	0.0

5.1 Setup

- **GMV_SUBS** Added arrays into GMV and GMVS buffers.

GMV(YT9%MGW)	transfer $gw^t(1:L)$ from P to C
GMVS(YT9%MGWTOP)	transfer $gw^t(0)$ from P to C.
GMVS(YT9%MCVWVTOPNL)	transfer ${\mathcal{M}_w}^t(0)$ from P to C

  Transfer of ${\mathcal{M}_w}^t(1:L)$ from P to C is assured by the array GMVS(YT9%MCVWVNL). This array is used in the traditional scheme as well and therefore it is not listed in the table.

- **YOMDYNA/NAMDYNA** Added LGWADV option.

- **SU0PHY** Protect computation for the option NDPSFI=1 as it is not consistent with the assumption done in *gw* scheme.

- **SUDIM2 and SUMP** Initialization of NPROMNH_GWADV=NPROMA resp. NPROMNH_GWADV=1 to save memory on the stack where the local variables are allocated in the case that *gw* scheme is not used.
- **SUDYN** The check of consistency with other model variables. Second order decentering doesn't work with the current implementation of *gw* scheme.
- **SUDYNA** Turn on/off the *gw* scheme LGWADV=.TRUE./.FALSE.. The check of consistency with other model variables.
- **SUSLB** Initialization of PB2 pointer. MSLB2VWVTOPT1 pointer is used to transfer final point tendency for *gw* equation at the top level, between CPG and CPGLAG.
- **YOMGEM ,SUALLO, SUVERT** Definition, allocation and initialization of VCUICOH and NVAUTH. Auxiliary arrays for interpolation weights computation.

5.2 Time stepping

The *gw* scheme specific computation are following

1. **CPG_GP**

   The vector of prognostic quantities $X = (\vec{v}, T, ln(\pi_s), \hat{q}, d)$ is transformed into the vector of prognostic quantities $Y = (\vec{v}, T, ln(\pi_s), \hat{q}, gw)$ at the beginning of grid point calculations.

   This transformation is done in the traditional scheme as well in order to evaluate RHS.

Routine	Purpose
GPGW	transform d into gw (PGWT0 or PGWT9)
GNH_TNDLAGADIAB_GW	compute nonlinear tendency of gw equation (PTNDGW)

2. **CPG_DYN/LACDYN**

   Preparation of the explicit step for all prognostic variables.

   The following formulation is implemented

   P: NESC $\tilde{Y}_F^{t+\Delta t(0)} = Y_O^t + \Delta t \frac{\mathcal{M}_w(Y^t)_F + \mathcal{M}_w(Y^t)_O}{2}$

   C(n): $\tilde{Y}_F^{t+\Delta t(n)} = Y_O^t + \Delta t \frac{\mathcal{M}_w(Y^{t+\Delta t(n-1)})_F + \mathcal{M}_w(Y^t)_O}{2}$

   LANHSI
   For the *gw* scheme there are two call of LANHSI from LACDYN. In the first call all the linear terms are set to zero. These are then given as arguments to LATTEX routine. During second call the linear terms are computed with d variable.

LANHSIB

The linear corrections are multiplied by time step and for the case of d_4 the $\mathcal{X}$ as added according to (13). After LANHSIB the PXSI arrays contains

P:	NESC	$PXSI = -\frac{\Delta t}{2}\mathcal{B}_d(X^t)$
C(n):		$PXSI = -\frac{\Delta t}{2}\mathcal{B}_d(X^{t+\Delta t(n-1)})$

and PVWVSI contains in the case of d_4 prognostic variable

for d_4:	P	$PVWVSI = PVWVSI + \mathcal{X}^t$
	C(n)	$PVWVSI = PVWVSI + \mathcal{X}^{t+\Delta t(n-1)}$.

This quantities are added into the computed explicit guess in CPGLAG/GPENDTR .

LATTEX and LATTES

The SL buffers are filled in the routines LATTEX and LATTES in the way as for traditional scheme (but all the linear terms are set to zero). There are two calls to LATTEX_DNT for the prognostic variable gw, because the top half level is computed independently. This is "reaction" on the fact that GMV structure can not contain half level quantities. Therefore the top levels quantities are artificially kept in the GMVS buffer.

The buffers are filled in the case of NXLAG=3 as

P:	NESC	$PXT1 = \frac{\Delta t}{2}\mathcal{M}_w(Y^t)$
		$PXL9 = Y^t$
		$PXL0 = \frac{\Delta t}{2}\mathcal{M}_w(Y^t)$
C(n):		$PXT1 = \frac{\Delta t}{2}\mathcal{M}_w(Y^{t+\Delta t(n-1)})$
		$PXL9 = Y^t$
		$PXL0 = \frac{\Delta t}{2}\mathcal{M}_w(Y^t)$

LAVABO

It is taken into account that the PVWVL0 and PVWVL9 SL buffers contain the half level quantity and the top half level buffer (index 0) can not be overwritten by 0.

3. **CALL_SL/LAPINEA**

   LAIDEPH

   The half level trajectories are computed.

   LAIDEPH/LARCINHA/ELASCAW

   The interpolation weigths for the half level quantities are computed.

4. **CALL_SL/LAPINEB**

   LAINORH/LARCINHB

   The half level quantities are interpolated.

The explicit guess is computed for the all prognostic quantities $\tilde{Y}^{t+\Delta t}$.

GNHGW2SVD
The explicit guess of $\tilde{gw}^{t+\Delta t}$ is transformed into the explicit guess of $\tilde{d_3}^{t+\Delta t}$. The transformation into variable $\tilde{d_3}^{t+\Delta t}$ is done also in the case that the d_4 is used (NVDVAR=4).

for d_3 and d_4 : $\quad \tilde{d_3}_l^{t+\Delta t} = \left(-\frac{\tilde{p}_l}{R_d T_l} \frac{\tilde{gw}_{\tilde{l}} - \tilde{gw}_{\tilde{l}-1}}{\delta \pi} \right)^{t+\Delta t}$

5. **CPGLAG/GPENDTR**

   Linear terms (in PXSI arrays) and $\mathcal{X}$ are added to the explicit guess and RHS is computed.

6 Tests

6.1 Warm and cold bubble test

The erroneous behavior has been reported for the traditional scheme for the test proposed by Robert,1995. This test is important when testing the NH kernels since it is nonlinear and in finite time the Lipschitz criteria is met in the lack of viscosity.

Test was run without sponge layer, without lateral coupling and without the horizontal and vertical diffusion (no viscosity). Results obtained with the traditional and gw scheme are shown on figure 1. The mesh size of tests is approx. 10m in the both directions. The time step is short $\Delta t = 1s$. We run in both cases 2TL ICI NESC scheme with one iteration.

More about this test can be found in the report of J. Masek (LACE stay report).

6.2 2D Trapped Lee wave test

To test stability we used the 2D trapped lee wave test of Bouttier, 2003, with 2TL PC NESC scheme with no diffusion, except the sponge layer at the top. The gw scheme has been found more stable than traditional scheme with d_4.

The both scheme performs almost identically with $\Delta t = 10s$ (figure 2) . When enlarging the time step twice $\Delta t = 20s$ the gw scheme remains still stable while traditional scheme blows after 72 time steps (figure 3). To stabilize the traditional scheme the horizontal diffusion have to be added (figure 4).

Similar behavior has been observed for the nonlinear non-hydrostatic flow regime over Agnesi shape mountain. The gw scheme gives the results similar (but not the same) to the traditional scheme for short time steps. gw scheme has been found to be more stable than the traditional scheme. This aspect has to be the subject of further investigation.

There are three potential reasons of erroneous behavior of traditional scheme:

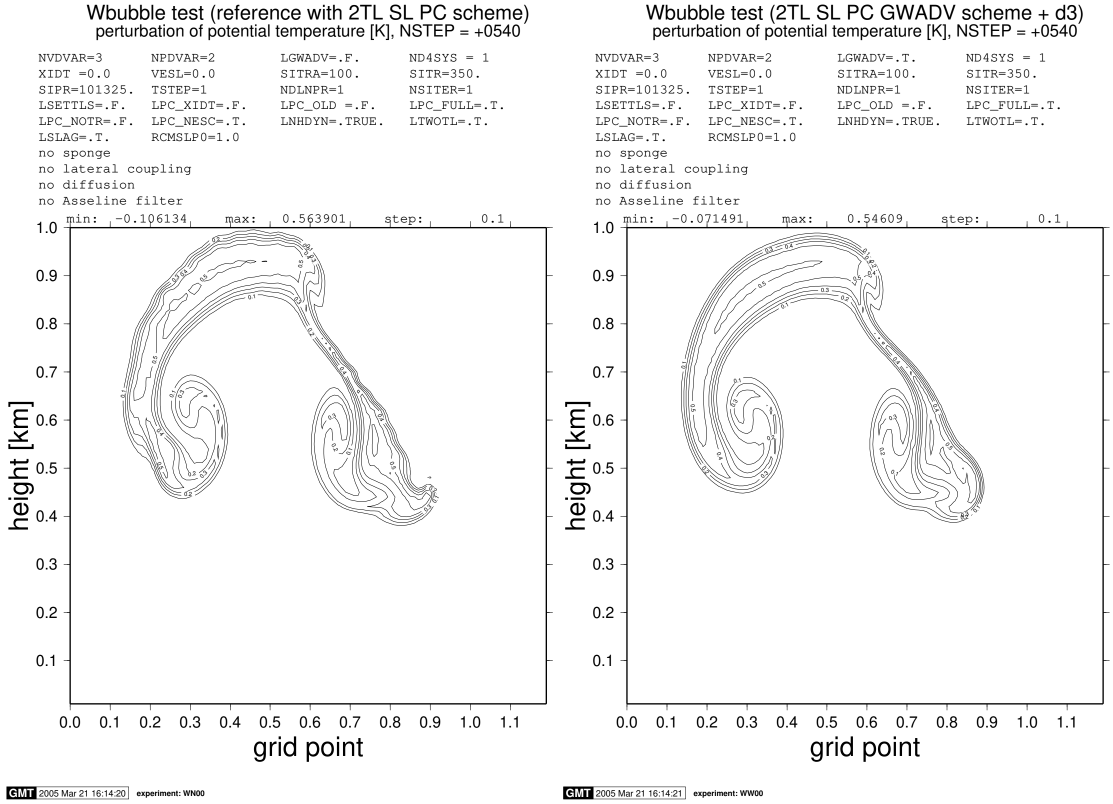

Figure 1: The state of warm and cold bubble after 9min of integration with $\Delta t = 1s$. The cold bubble is not visible. The potential temperature perturbations are plotted with isoline 0.1K. The traditional scheme result (left) is noisy and the simulation blows before it reaches 10min integration length. The *gw* scheme result (right) is correct and the simulation can be integrated to the end (10min) without additional problem.

1. *gw* scheme has "unexplained" diffusive properties, comparing to traditional scheme. Almost the same performance of both schemes with $\Delta t = 10s$ points against this reasoning.

2. traditional scheme has stability and accuracy problems for the regimes close to Lipschitz criteria (long Δt). The high-frequency noise is produced in such regimes and the scheme becomes unstable. Additional diffusion have to be applied to remove this noise and stabilize the scheme. This explanation is also supported by the warm and cold bubble test.

3. there is still unsolved stability problem in traditional scheme with d_4, related to pseudo-prognostic equation (choice ND4SYS).

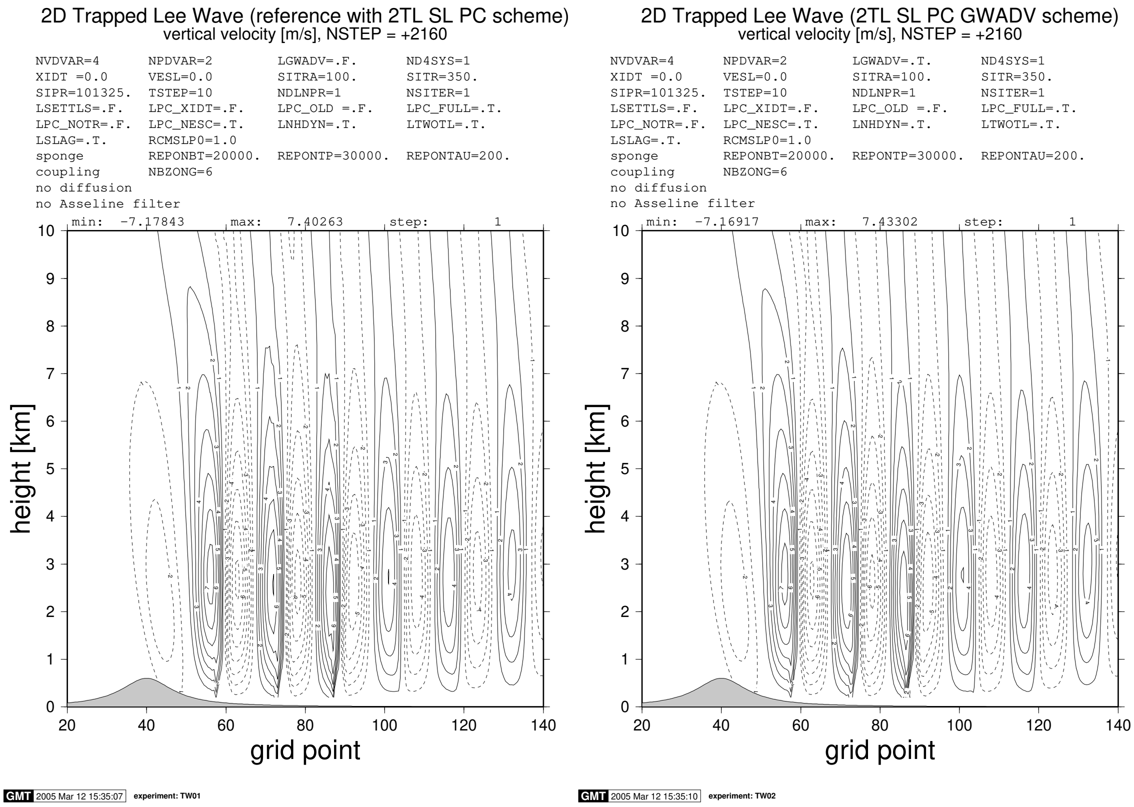

Figure 2: The Trapped Lee wave test with $\Delta t = 10s$. Vertical velocity array plotted with isoline $1ms^{-1}$. The traditional scheme results (left) are almost identical to the *gw* scheme results (right)

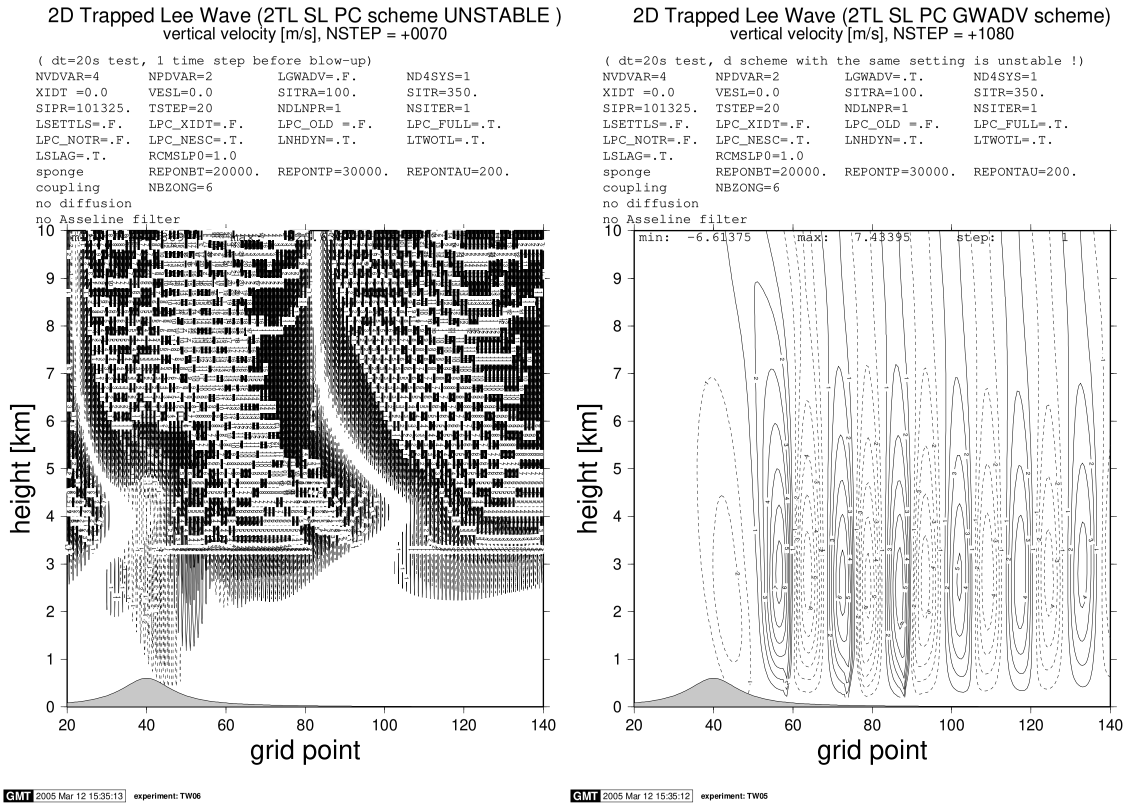

Figure 3: The Trapped Lee wave test with $\Delta t = 20s$. Vertical velocity array plotted with isoline $1ms^{-1}$. The traditional scheme results (left) inhibits severe instability with the center located in the sponge layer. The gw scheme results are still correct (right)

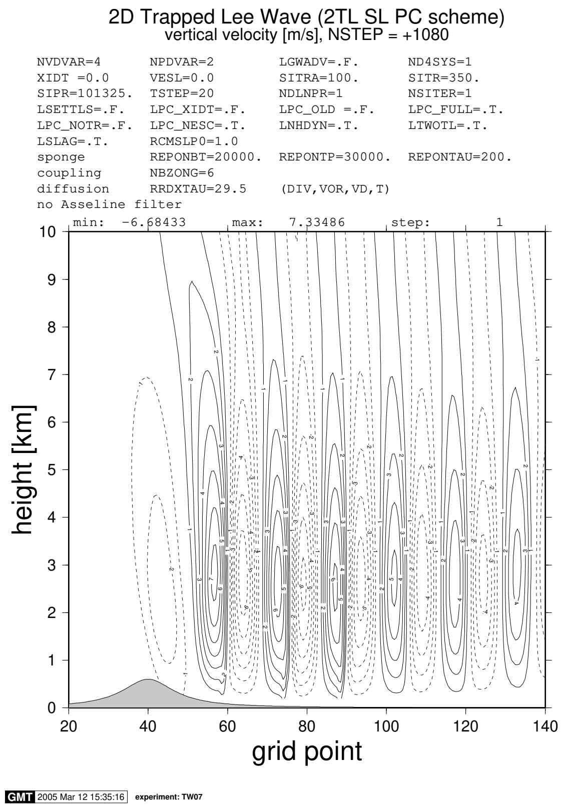

Figure 4: The Trapped Lee wave test with $\Delta t = 20s$. Vertical velocity array plotted with isoline $1ms^{-1}$. The traditional scheme can be integrated only with the horizontal diffusion and the result is on the figure.