Vertical finite element discretisation in NH kernel of model system AAA

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1 Introduction

We have performed following activities during 4 weeks period

- phasing of development into export version 40t1_bf5

  Before arrival of Spanish colleagues we needed to phase our development into latest export version of model available at the moment. we needed to be prepared for further merge with development of Alvaro Subias based on 40h1 version. The parts related to dynamics are identical in both reference version and therefore they could be than easily phased.

- phasing of HIRLAM and LACE development

  After arrival of HIRLAM colleagues (Alvaro Subias and Juan Simmaro), we agreed that we will produce common code. We have phased our development together and validated it.

  We validate our code against the 2D idealized tests reported in previous report [1]

- work on article to be posted to MWR

  Juan Simmaro took the responsibility to further improve the draft of our common paper (he is an co-author). We discussed issues continuously and the new, improved version of paper has been drafted. The article will be send into Monthly Weather Review. Me (Jozef Vivoda) as first author must start to ensure financial aspects of publication.

- remaining issues reported in 2015
  - instability of operators with odd spline order (3,5,7 ...).
  - implementation of half level gw - $LVFE\_GW\_HALF$ with VFE treatment of transformations between d and gw
  - instability of $LVFE\_GW$ and $NITERHELM > 0$

    This was reported, but with currect version of code, the instability was not reproduced.

- noise at the boundaries of operators

  During our discussion about the properties of operators, we have identified the problem with noise of operators at the model boundaries.

  The noise appeared consistent of HIRLAM and in our version of operators. Our version of operators construction used two matrix inversions (projection operator from grid point to VFE space, stiffness matrix inversion), while HIRLAM version uses only one inversion (projection). Therefore we concluded that he noise origins in inversion if ill-posed matrix. We therefore decided to investigate this issue more in detail (see further).

  comments: Noise in integral operator can be suppressed be smoothing definition of η on half levels computed under key $LVFE\_REGETA$.

  Operators with BCs imposed on derivatives are more sensitive to noise appearance that those with BCs imposed on values.

- fix of operators in 40t1 export version

  When we integrate constant function $f = 1$ with correctly defined VFE integral operator and well prescribed BCs it must be valid

$$\int_0^\eta 1 d\eta = \eta$$

  We found this condition not to be exactly fulfilled in 40t1. This problem was fixed.

- definition of knots and explicit values of η on model levels

  We adopted procedure, that is superior to those adopted in previous periods.

  Any definition of η that satisfies the conditions $\eta_{l-1} < \eta_l$ and $\eta_{\tilde{l}-1} < \eta_l < \eta_{\tilde{l}}$ is correct. However, not all definitions provides the VFE scheme with good stability properties of time stepping.

  The main idea is to determine the first guess of knots t. We than construct the B-spline basis of order $C-1$ and we define the cost function

$$J = \sum_{l=0}^{L} \left(\eta_{\tilde{l}} - \xi_l\right)^2 .$$

  It measures the distance between position of "special" l-th spline point ξ_l and model half level $\eta_{\tilde{l}}$. We determine knot sequence t such that it minimize the given cost function J. Special spline points ξ_l are determine as position of maxima of l-th spline function (LVFE_MAXIMA=.T.) or as a Greville abscissa (LVFE_MAXIMA=.F.). Once we minimize J the new values of half level $\eta_{\tilde{l}} = \xi_l$.

  Further we construct the basis of order C. The "special" points define the full level $\eta_l = \xi_l$. The "special" points are defined in the same manner as in the previous paragraph.

To solve the nonlinear minimization problem for J we exploited MINPACK package routine LMDIM1 (SUVERT).

The details of the procedure are described in these steps:

1. first guess of half levels η (LVFE_REGETA=.F.)

$$\eta_{\tilde{l}} = \frac{A_{\tilde{l}}}{{\pi_s}^*} + B_{\tilde{l}}$$

   This definition is used due to property $m^* = \frac{\partial \pi^*}{\partial \eta} = const.$. This property is important in definition of A and B on full levels. See next paragraph.

2. first guess of full level η

$$\eta_l = \frac{\eta_{\tilde{l}} + \eta_{\tilde{l}-1}}{2}$$

3. first guess of internal knot sequence τ

   This guess is different for even and odd order of bases (defined by NVFE_ORDER in fort.4), For even order (2,4,6,...) first guess of τ is constructed from values of η at model full levels, and for odd order (3,5,7,...) first guess is defined from values of η at model half levels.

   The internal knots τ are as follows for order of splines $C = 2, 4, 6$

$$\tau_l = \eta_{l+1}, l = 1, L + 2 - C,$$

   and for odd order of splines $C = 3, 5, 7...$

$$\tau_l = \eta_{\tilde{l}+1}, l = 1, L + 2 - C.$$

   Here L is number of full model levels consistent with NFLEVG.

4. construction of B-spline basis of order $C - 1$ and determination of half level η

   Internal knots τ are completed by multiple knots $C - 1$ at boundaries and the B-spline basis is constructed. The number of B-spline functions is $L + 1$, the same as number of model half levels. We determine either position of maxima of each spline function, or we compute Greville values directly from knot sequence (LVFE_MAXIMAS=.T. resp. .F.).

   Maxima of l-th function is computed using Newton's bisection iterative algorithm (SUVFE_CPSPLINES). The n-th iteration for l-th function yields

$$\xi_l^{n+1} = \xi_l^n - \frac{B_l'(\xi_n)}{B_l''(\xi_n)}.$$

   We stop when $\frac{B'(\xi_n)}{B''(\xi_n)} < 10^{-5}$. From the properties of B-spline basis function constructed by DeBoor algorithm is valid $\xi_0 = 0$ and $\xi_L = 1$. It is consistent with the values of half level η.

   The Greville abscissa is defined as

$$\xi_l = \frac{\sum_{i=1}^{C-2} t_{l+i}}{C-2}.$$

Here full knot vector is used $t_1 = ... = t_{C-1} = 0, \tau, t_{L+2} = ... = t_{L+C} = 1$. Due to multiplicity of knots at boundaries, it is valid $\xi_0 = 0$ and $\xi_L = 1$. This is consistent with the values of half level η.
We define cost function J

$$J = \sum_{l=0}^{L} \left(\eta_{\tilde{l}} - \xi_l\right)^2,$$

and we look for internal knot vector τ such that minimize cost function J. We finish iterations when $J < 10^{-5}$.

The values of half levels η are then determined as final values of ξ.

The internal knot vector τ is stored (VFE_KNOTS in SUVERT) and used during the construction of knots inside VFE operators ($SUVERTFEB->SUVFE\_KNOTS$).

Minimization of nonlinear mean-square method is performed in SUVERT by MINPACK routine LMDIM1.

5. construction of B-spline basis of order C and determination of full level η

   The same method is applied as in the previous paragraph, except that the minimization of cost function is not performed. We construct B-spline basis of order C using internal knot vector τ as $t_1 = ... = t_C = 0, \tau, t_{L+3} = ... = t_{L+2+C} = 1$.
   We can show that there is $L+2$ B-spline basis functions (NFLEVG + 2 material boundaries).
   We determine $L+2$ "special" values ξ as in the previous section (maximas or Greville points) and we set full level η to be equal to ξ points.
   Greville points in this case are computed as

$$\xi_l = \frac{\sum_{i=1}^{C-1} t_{l+i}}{C-1}.$$

- LVFE_APPROX - approximation vs. interpolation with B-spline basis

  We have adopted Greville relation between knots and full levels values, because this definition is mandatory for Schoenberg variation diminishing approximation of sampled points (η_l, f_l) of unknown function f. The Schoenberg algorithm assumes that the full levels are computed from knots using Greville relation. The spline approximation is then

$$S(\eta) = \sum_{l=0}^{L+1} f(\eta_l) B_l(\eta).$$

The VFE coefficients are equal to values of function in sampled points $f(\eta_l) = f_l$. The projection from GP space to VFE space is therefore identity matrix. This is implemented in the code under key LVFE_APPROX.

Schoenberg VDA algorithm provides non-oscillatory approximation of function f. It preserves shape (in the case of cubic spline, it preserves sign, monotonicity and convex properties of line that connect linearly sampled points (η_l, f_l)).

Using of LVFE_APPROX provides more stable and less noisy solution then interpolating polynomial, however theoretically less accurate than interpolating spline. But we have to consider, that approximating approach provides still more accurate results than FD method in vertical.

- definition of m^* and full level A and B

  Here we start from the definition defined in Hortal and Untch. We adjusted it to keep exact property $m^* = \frac{1}{{\pi_s}^*}\frac{d\pi^*}{d\eta} = 1$. Here we assume that ${\pi_s}^* = \pi_{ref}$, with π_{ref} pressure used in scaling of A (RP00 namelist parameter is set to the same value as SIPR).

  1. we compute depth of layers $\delta\eta_l$ and $\delta\pi_l^*$ from half level quantities as $\delta\eta_l = \eta_{\tilde{l}} - \eta_{\tilde{l}-1}$ and $\delta\pi_l^* = \delta A_l + \delta B_l {\pi_s}^*$. Here $\delta A_l = A_{\tilde{l}} - A_{\tilde{l}-1}$ and $\delta B_l = B_{\tilde{l}} - B_{\tilde{l}-1}$.

     Further δA and δB profiles must be adjusted to satisfy

     $$\int_0^1 \frac{\delta A}{\delta\eta} d\eta = 0$$

     and

     $$\int_0^1 \frac{\delta B}{\delta\eta} d\eta = 1,$$

     when computed with VFE integral operator defined in next step.

  2. VFE operators are defined using internal knots vector τ computed in previous section
  3. adjustment of δB

     Integral of δB over whole domain gives

     $$\int_0^1 \frac{\delta B}{\delta\eta} d\eta = a.$$

     To ensure above integral is equal to 1, we rescale $\delta B = \frac{\delta B}{a}$.
  4. adjustment of δA

     We require $\frac{m^*}{{\pi_s}^*} = 1$. It is exactly valid with adopted definition of $\eta = \frac{A}{{\pi_s}^*} + B$. However, previous adjustment of η during minimization of J and adjustment of δB has changed this property slightly We redefine profile of δA to recover lost property as

$$\delta A_l = \delta\eta_l {\pi_s}^* \left(1 - \delta B_l\right).$$

Due to fact that $\int_0^1 \frac{\delta B}{\delta \eta} d\eta = 1$ and $\int_0^1 \frac{m^*}{{\pi_s}^*} d\eta = 1$ it can be shown that new definition of δA satisfy exactly

$$\int_0^1 \frac{\delta A}{\delta \eta} d\eta = 0.$$

5. full level of A and B

   Integrating δA and δB from model top to given level η, we can compute full level values of A and B. Due to property $\int_0^\eta \frac{m^*}{{\pi_s}^*} d\eta = \eta$ we see that relation between full level η and full level A and B is

$$\eta_l = \frac{A_l}{{\pi_s}^*} + B_l.$$

   This property is valid for every reasonable definition of full levels. It is satisfied by the property of VFE operator $\int_0^\eta 1 d\eta = \eta$.

- oscillation in invertible operators

  The principal problem in definition of invertible VFE operator is that invertibility requires fixed knots sequence. The input and output B-slines basis function have different order. This means that the basis function's maximas are located on different locations. When integral, derivatives and functions are located on full levels, then we can not guarantee diagonal dominance of projection operator in both invertible operators.

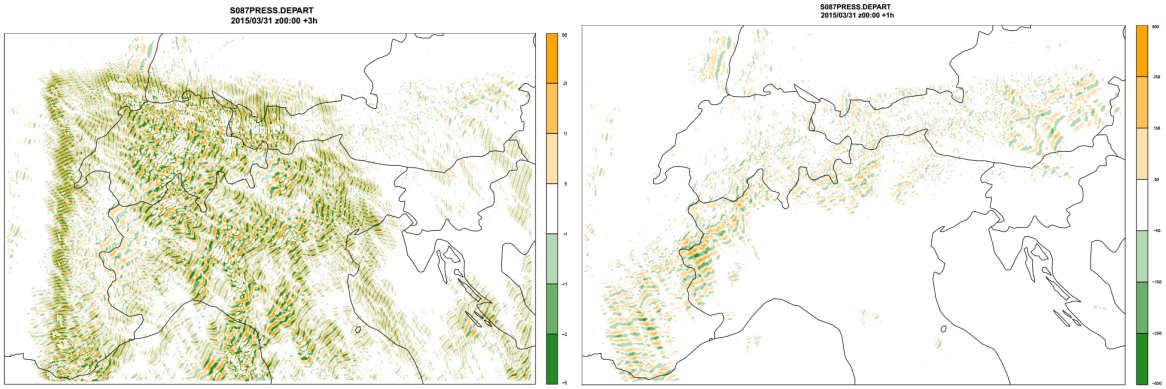
  Therefore the idea is to use staggering of gw also in VFE scheme. The distribution of full and half levels would be defined from maxima's of B-spline functions (even order - definition of full levels, associated odd order - definition of half levels).

- topics out of VFE scheme - noise in SL scheme and divergence of trajectory search algorithm

  It was reported that LPC_FULL scheme with reiteration of SL trajectories produce noisy solution. We have confirmed these results. We found that PC scheme where SL trajectories are computed only during predictor step - LPC_CHEAP is noise free. It was not the aim of stay to investigate this problem. Finding is an important topic for further investigations and shall appear in LACE plane for near future.
  As we increase model resolution to kilo-metric scales, the local divergence can increase significantly. Therefore we could locally break Lipschitz criteria with long time steps and the trajectory search algorithm is divergent. We should investigate this aspect of SL scheme in very high resolution NH models.

  In order to increase accuracy of SL trajectories in highly varying divergence wind fields, we have to increase number of iterations in SL trajectory search in high-res model. However this is impossible when the trajectory search is divergent.

  Similar problem was identified in ECMWF model and fixed by local change of computation of half level wind (they avoid using extrapolated wind in such case) [2].



(a) *PC_FULL* (b) *PC_CHEAP*

Figure 1: NH model with ALARO-1 above Alpine region with 1.25km resolution and 87 vertical levels. Pressure departure at lowest model level is plotted. The noise in LPC_FULL scheme is apparent.

The noise is shown on Figure 1. There appear the noise in PC_FULL scheme during whole integration, while PC_CHEAP scheme was noise free.

We found that the intensity of noise with LPC_FULL is sensitive to the choice of $SITR$ in this case (SITR=250 removed noise in the same way as LPC_CHEAP did). This was out of our topic, and shall be further investigated.

References

[1] Jozef Vivoda, *Vertical finite element discretisation in NH kernel of model system AAA*, RC LACE Stay report, Prague, 2015.

[2] M. Diamantakis and L. Magnusson, *Numerical sensitivity of the ECMWF model to Semi-Lagrangian departure point iterations*, ECMWF Technical Memoranda, 2015 (http://www.ecmwf.int/sites/default/files/elibrary/2015/15302-numerical-sensitivity-ecmwf-model-semi-lagrangian-departure-point-iterations.pdf)