Application of ENO technique to semi-Lagrangian interpolations

RC LACE stay report
Scientific supervisors: Ján Mašek and Petra Smolíková

Alexandra Crăciun
NMA, Romania

CHMI, Prague 19.05-13.06.2014

1 Introduction

The ALADIN model uses a semi-Lagrangian advection scheme. This scheme employs spatial interpolations and its properties are strongly affected by choice of different interpolator.
The purpose of this study was to explore some alternative interpolators that are less overshooting than commonly used cubic Lagrange polynomial but still accurate enough. Because of the difficulty in increasing the number of points in the stencil we restricted to types of interpolation that use at most 4 point stencil available in the current code. Of course, the overshootings of the cubic Lagrange interpolator can be removed using the quasi-monotonic correction. On the other hand, if we apply quasi-monotonic treatment globally, we loose accuracy. In order to avoid this, the solution could be one that applies quasi-monotonic version only in the vicinity of discontinuities, but not in the smooth part of the function.
A way to remove the overshootings is the ENO technique. Some preliminary tests were made by Ján Mašek, implementing this method in a toy system - 1D nonlinear advection scheme. The task was then to evaluate the behaviour of the ENO interpolation in some 2D experiments. After doing this, the next step was to try the weighted variant of this technique (WENO) and another type of weighted combination of linear and cubic interpolators.

2 Experiments

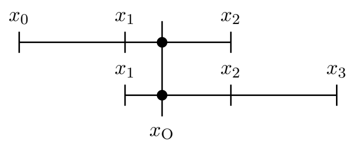
Implementation in the cycle 38t1

For testing this scheme, research version of the code was created, with computation of interpolation weights directly in subroutine LAITRI, restricted to LAM case with LREGETA=.T. (i.e. assuming regular nodes both in horizontal and in vertical). Two dimensional interpolation subroutine LAIDDI was adapted accordingly. These subroutines contain also the modifications required for other interpolation schemes tested during the stay.
There, the variable ξ used in equation (2) is either PDVER (for vertical interpolation), PDLAT (in latitude) or PDLO (in longitude). To get all needed arrays, we have changed LAITRE_GFL, LAITRE_GMV, LARCINA, LARCINB and LARCIN2 as well.
The necessary switch to enable ENO, was created in the new module YOMENO and declared in the namelist namdyn.h. The name of the general switch for this interpolator is LENO; other keys for switching the ENO modification on for distinct variables are not needed. In a similar way, the logical variable LWENO allows the use of WENO interpolation, if LENO is also set true. (see subsection 2.2). As to apply the mixture between cubic and linear in-

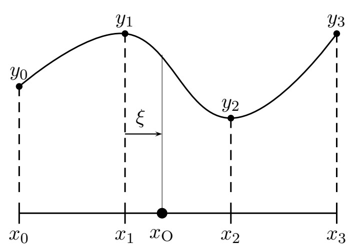
terpolation (see subsection 2.3) it is enough to switch the LVAR key in the namelist NAMDYN and set a value for the parameter RALPHA. All modifications are available in Prague on *yaga* machine under the TUC branch Arp_mma130_CY38t1_eno.

2.1 ENO scheme

For a second order interpolation scheme, a 3 point stencil is used to construct a 1D quadratic interpolator. Having four nodes and the interpolation point in the central interval, we can choose either the left or the right 3 point stencil to get the interpolated value. If this choice is made based on the smoothness of the solution we speak about the ENO technique. In particular the choice between the stencils is made by computing the finite difference aproximation for second derivate for each of the two stencils and using the one with smaller absolute value of that quantity.



Here, x_i are grid point coordinates with corresponding function values y_i. For example: if $|y_2 - 2y_1 + y_0| < |y_3 - 2y_2 + y_1|$ (see the figure below),



we will use the quadratic interpolator on stencil $0-1-2$ and the interpolating polynomial will have the form:

$$y(x_O) = y_0 \frac{(x_O - x_1)(x_O - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x_O - x_0)(x_O - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x_O - x_0)(x_O - x_1)}{(x_2 - x_0)(x_2 - x_1)} \tag{1}$$

For regular mesh size Δx we get:

$$y(x_O) = y_0 \frac{\xi(\xi - 1)}{2} + y_1 \frac{(\xi + 1)(\xi - 1)}{-1} + y_2 \frac{(\xi + 1)\xi}{2}, \tag{2}$$

where $\xi = \frac{x_O - x_1}{\Delta x}$.

2D academic experiments

In order to evaluate the results we have used three tests based on the classical Robert bubble test:

- warm (+0.15K) and cold (-0.5K) bubbles with smooth boundary in the field of constant potential temperature (300K)
- the same bubbles advected with the wind speed of 2m/s
- warm bubble (+0.5K) with sharp boundary in the field of constant potential temperature (300K) advected with the wind speed of 2m/s.

The reference solution uses the quadratic interpolator.

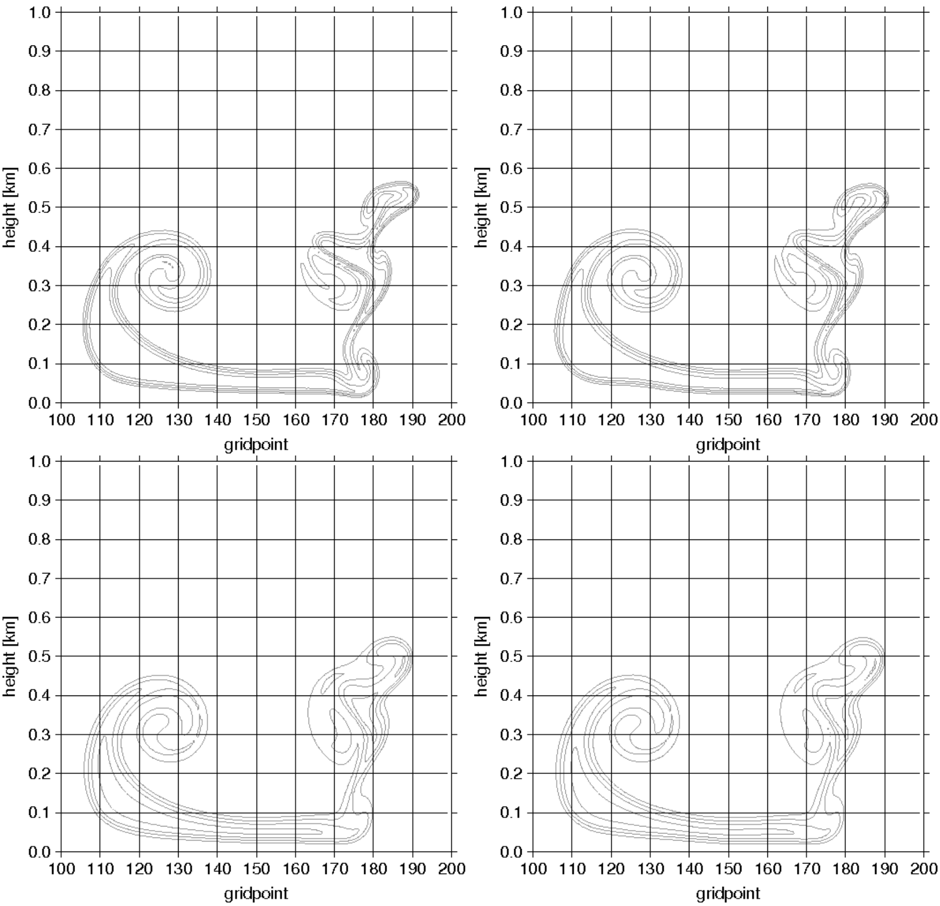


Figure 1: Bubble test – perturbation of potential temperature after 10 min: quadratic interpolator (top left), quadratic with ENO stencil (bottom left) and their quasi-monotonic versions (right column)

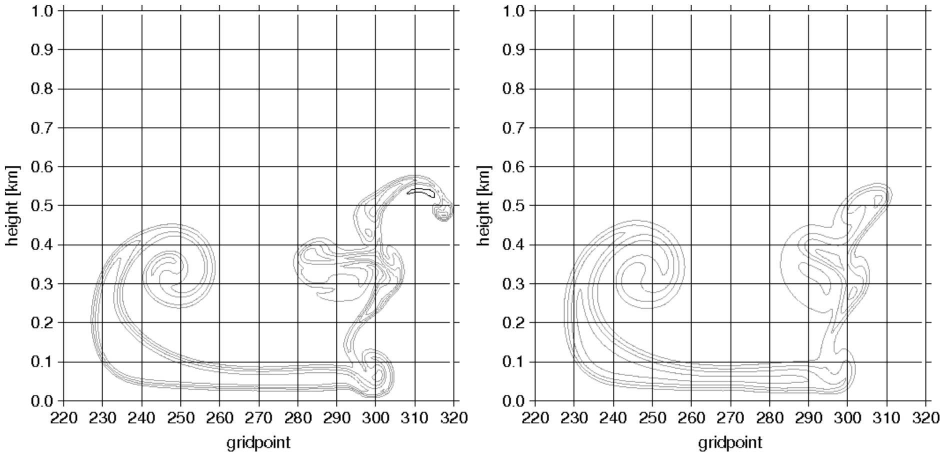


Figure 2: Advected bubble – perturbation of potential temperature after 10 min: quadratic interpolator (left) and using ENO stencil (right)

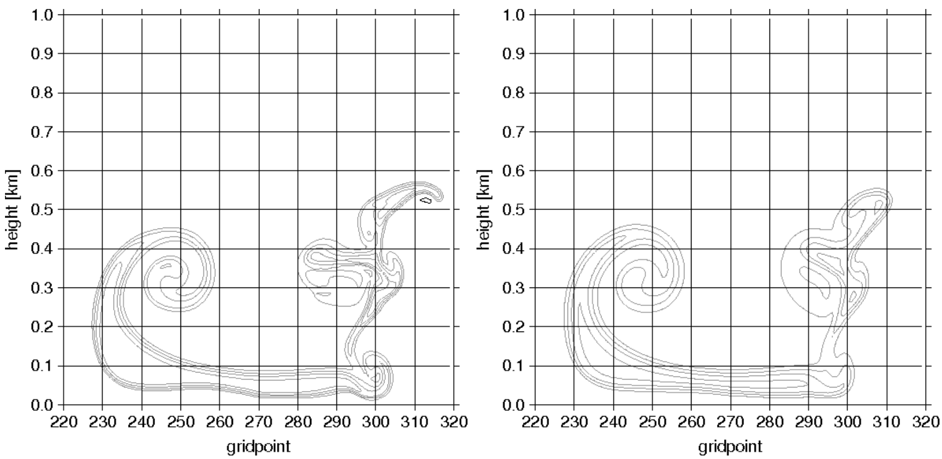


Figure 3: Advected bubble – perturbation of potential temperature after 10 min: quasi-monotonic versions of quadratic interpolator (left) and using ENO stencil (right)

In figure 1, we can see that when applying ENO the solution becomes much smoother and the gradients are less sharp. Also, the quasi-monotonic treatment doesn't lead to much different solution because using ENO already removes many of the details in the bubble plot (figure 2). In the case of advected bubble and quasi-monotonic treatment, we can observe in the bubble test (figure 3) that the quadratic interpolator produces some distorsions in the bottom part of the bubble in contrast to the one applied on ENO stencil.

2.2 WENO scheme

WENO is the weighted variant of the ENO scheme. In the second order WENO scheme, instead of choosing between two stencils, we use a weighted combination of the two 3 point stencils.

$$y = p_1 \cdot w_1 + p_2 \cdot w_2, \quad w_1 + w_2 = 1 \tag{3}$$

with $w_1 + w_2 = 1$ Here, p_1 is the interpolated value on the first stencil and p_2 the value on the second stencil. In the ENO interpolation case, this weights would be either 0 or 1, depending on the stencil used for interpolation. The way of choosing these weights for the two polynomials is not strictly defined. Here, the weights were chosen as:

$$w_1 = \frac{1}{2} + 4\left(\frac{S_1}{S_1 + S_2} - \frac{1}{2}\right)^3 \tag{4}$$

$$w_2 = \frac{1}{2} - 4\left(\frac{S_1}{S_1 + S_2} - \frac{1}{2}\right)^3 \tag{5}$$

where:

$$S_1 = |y_2 - 2y_1 + y_0| + \varepsilon, \quad S_2 = |y_3 - 2y_2 + y_1| + \varepsilon \tag{6}$$

(ε is a small number being used just for numerical safety, to prevent division by 0).

This method has been tested in 2D academic experiments with cold and warm bubbles (as described in subsection 2.1). Figure 4 shows that for the WENO combination interpolation, the bubble contains much of the details contained in the bubble result for the quadratic interpolator, still with less sharp gradients. As for the case of advection (figure 5), we can see that the WENO interpolation is smoother and less noisy then the quadratic interpolator (similar to ENO , figure 2 and 3). Also, the deformation that appears when applying advected and quasi-monotonic versions is still present for the two interpolators.

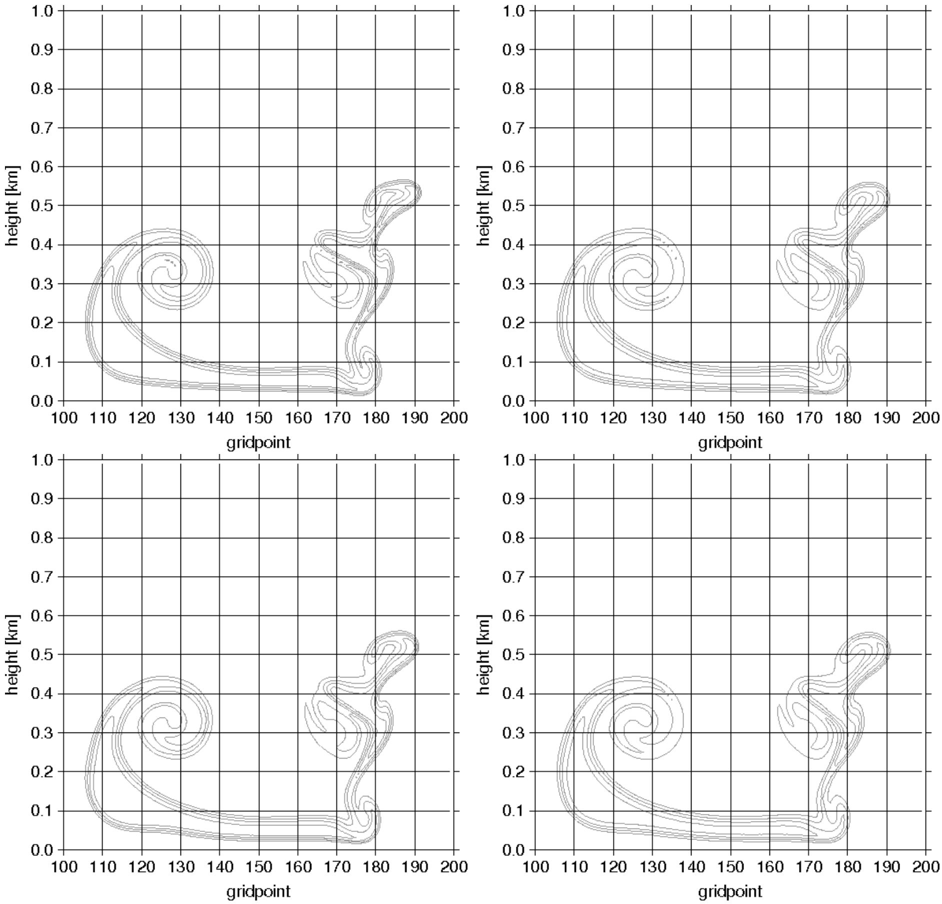


Figure 4: Bubble test – perturbation of potential temperature after 10 min: quadratic interpolator (top left), quadratic with WENO combination (top right) and their quasi-monotonic versions (second row)

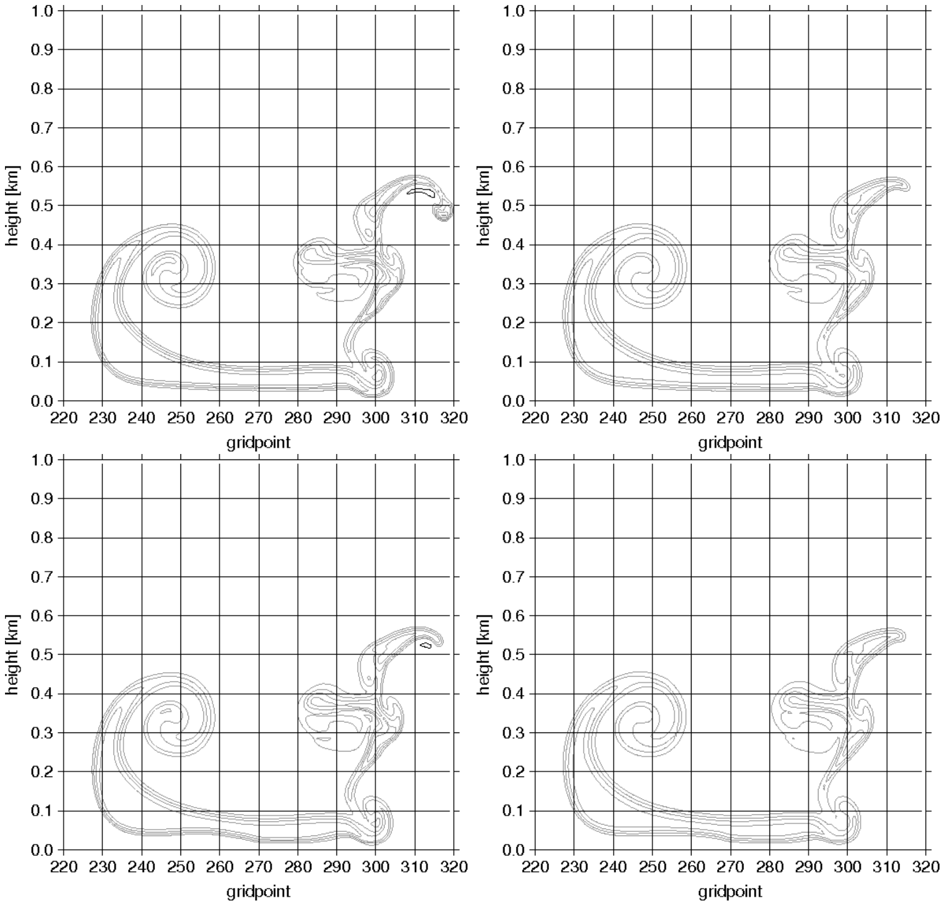


Figure 5: Advected bubble – perturbation of potential temperature after 10 min: quadratic interpolator (top left), quadratic with WENO combination (top right) and their quasi-monotonic versions (second row)

2.3 Mixture of cubic Lagrange and linear interpolator

Another method for semi-Lagrangian interpolations is one that uses variable accuracy, the interpolated variable being a weighted combination of cubic Lagrange and linear interpolator, with the purpose of removing the overshoots in the vicinity of discontinuities by using non-overshooting linear interpolator there, while keeping accurate cubic Lagrange interpolator in smooth part of the solution. The interpolated variable has the following form:

$$p = p_c \cdot (1 - W) + p_l \cdot W, \tag{7}$$

where p_c is the cubic interpolator and p_l the linear interpolator,

$$W = w^2 \cdot (3 - 2w), \tag{8}$$

$$w = \Big(\frac{3}{2}\frac{\max(|y_1 - y_0| + \varepsilon, |y_2 - y_1| + \varepsilon, |y_3 - y_2| + \varepsilon)}{|y_1 - y_0| + |y_2 - y_1| + |y_3 - y_2| + 3\varepsilon} - \frac{1}{2}\Big)^\alpha \tag{9}$$

The term in bracket in equation (9) is used as smoothness indicator, approaching 1 when total variation of y is dominated by its change in single interval and being 0 when y changes linearly; α is a tuning exponent which controls the amount of linear interpolator, in shown experiments value $\alpha = 2$ was used. Using higher α would give results closer to cubic Lagrange interpolator, while lower α would give smoothed solution with less overshoots. Also, the manner of choosing between cubic Lagrange and linear interpolation could use different formula.

Looking at the figures below (figure 6 and 7), we can see that this kind of variation between cubic and linear interpolation gives a solution sharper than ENO interpolation (figure 1) and much similar to WENO (figure 4) in the bubble test.

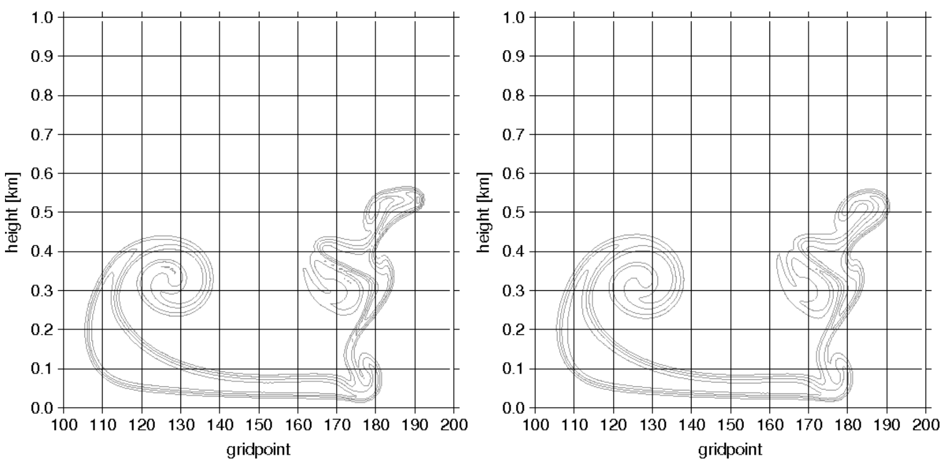


Figure 6: Bubble test – perturbation of potential temperature after 10 min: quadratic interpolator (left), cubic Lagrange/linear variation (right)

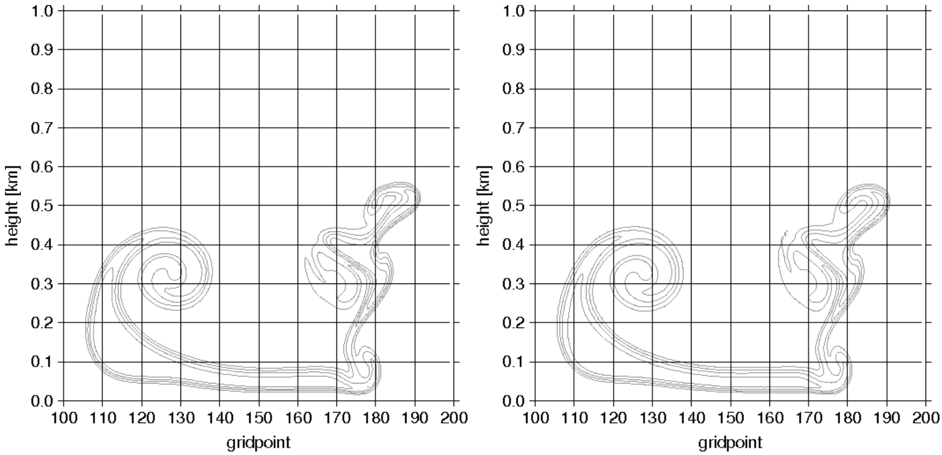


Figure 7: Bubble test – perturbation of potential temperature after 10 min, quasi-monotonic versions of: quadratic interpolator (left), cubic Lagrange/linear variation (right)

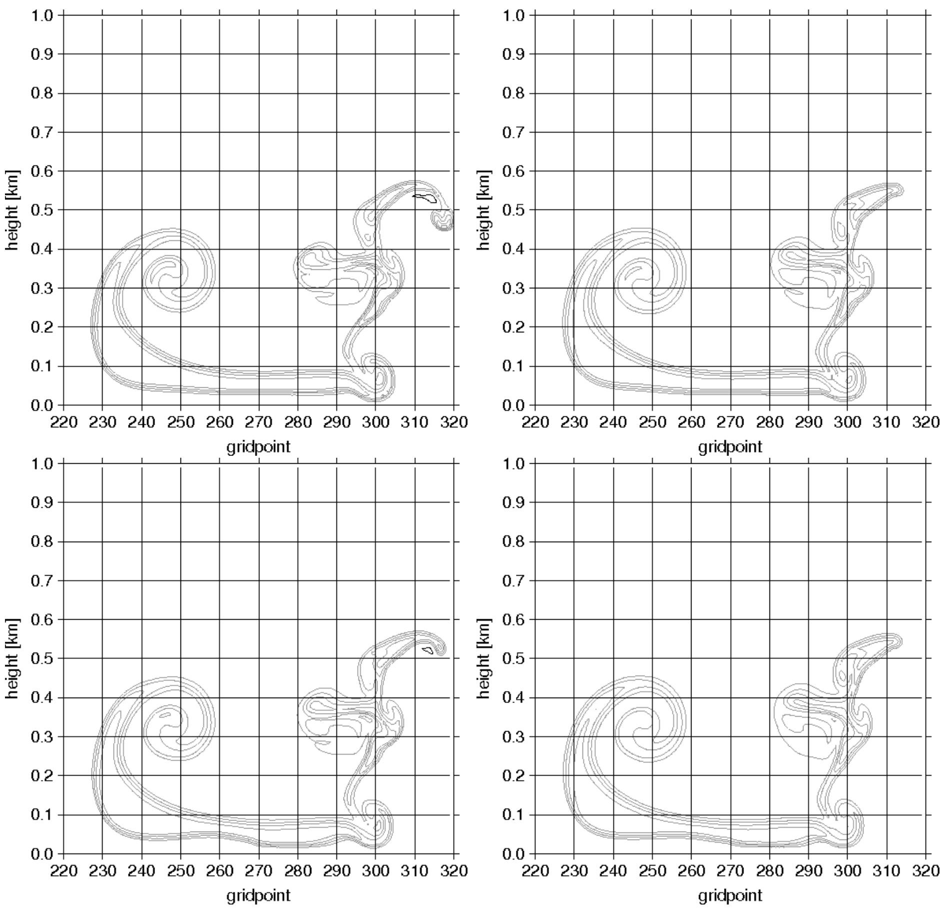


Figure 8: Advected bubble – perturbation of potential temperature after 10 min: quadratic interpolator (top left), cubic Lagrange/linear variation (top right) and their quasi-monotonic versions (second row)

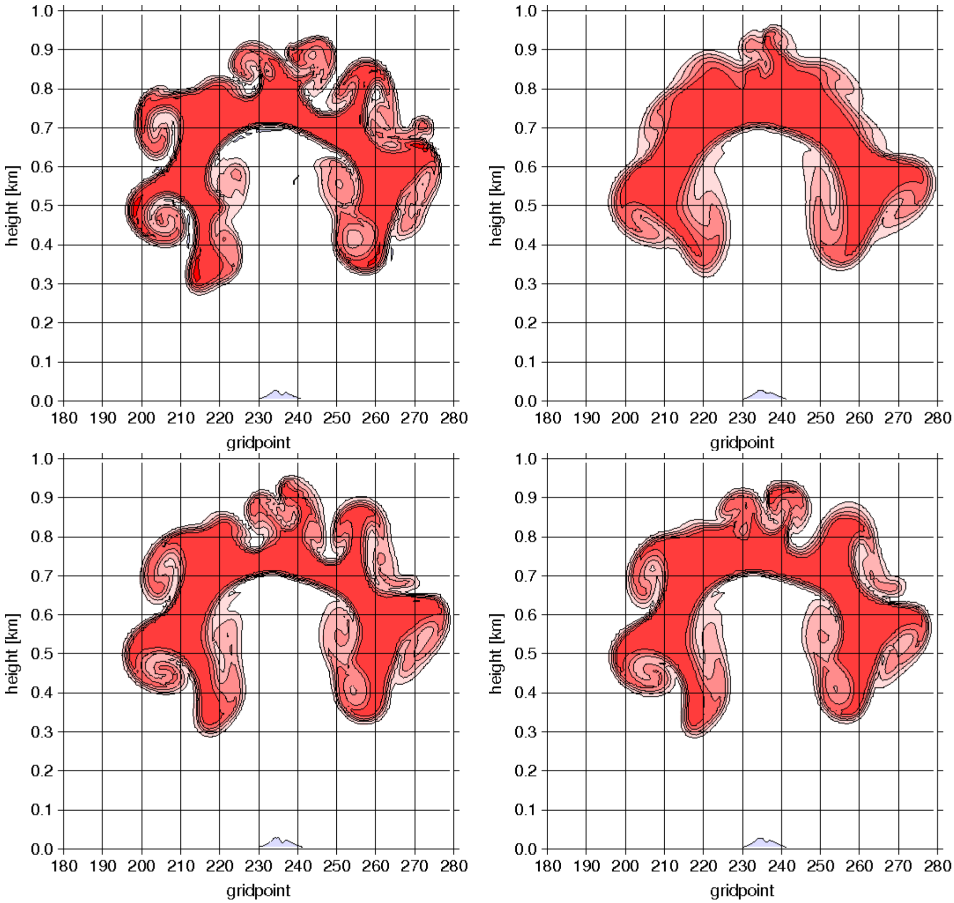


Figure 9: Sharp warm bubble test with advection – perturbation of potential temperature: quadratic interpolator (top left), quadratic with ENO stencil (top right), cubic Lagrange/linear variation (bottom left) and quadratic with WENO combination (bottom right)

It is obvious from the plots obtained with the sharp warm bubble test (above) that the most smoothing interpolator is the one with ENO stencil, while the solution for the quadratic and WENO combination has more sharp gradients and is very similar to the one for cubic Lagrange and linear combination interpolator. On the other hand, we know that the analytical solution is symmetric, so that asymmetry on the plots in figure 9 is distortion due to background advection. Most symmetric solution is that of quadratic ENO, but this is at the expense of lost sharpness.

These results may be confirmed in the 1D linear advection test. For this purpose we chose an advection of a sine wave with the wavelength $20\Delta x$ to demonstrate the behavior of interpolation technique on smooth functions and linear advection of a rectangular pulse to demonstrate the behavior close to discontinuities. The domain is periodic with the initial state being plotted in red, and with different colors used for the solution after several time periods.

– initial profile
– advected profile after 1 revolution
– advected profile after 2 revolutions
– advected profile after 3 revolutions
– advected profile after 4 revolutions
– advected profile after 5 revolutions

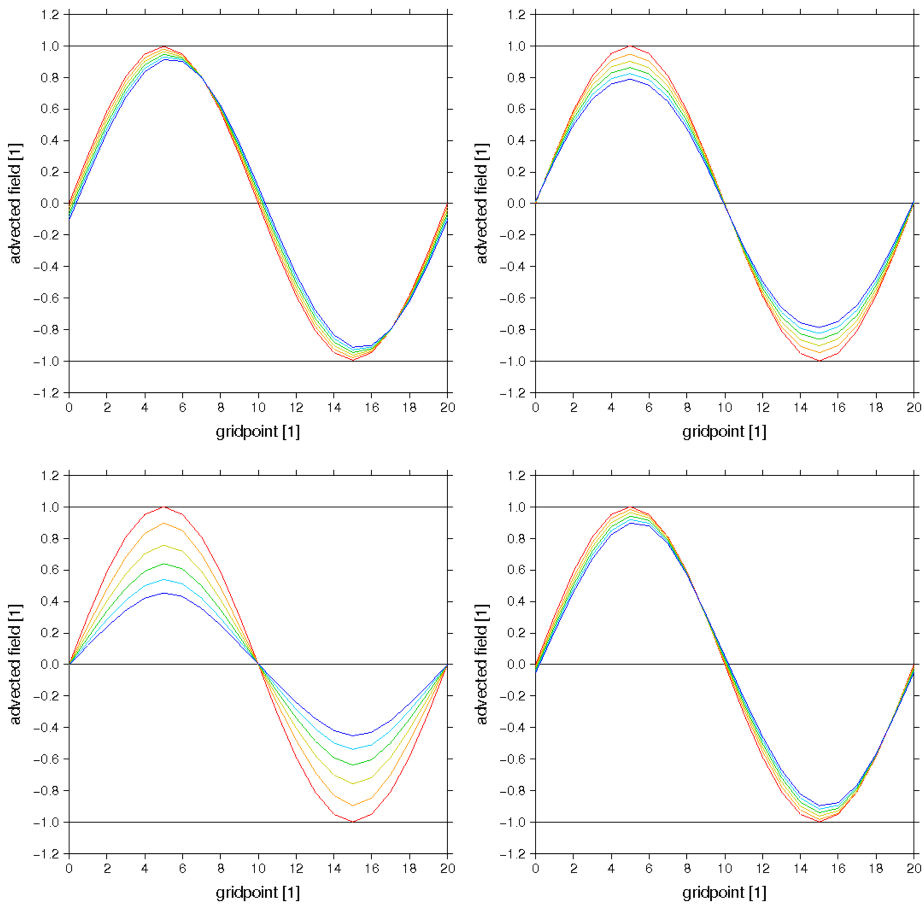


Figure 10: Linear advection of sine wave in periodic domain (CFL number = 0.2): quadratic interpolator (top left), quadratic with ENO stencil (bottom left), cubic Lagrange/linear variation (top right) and quadratic with WENO combination (bottom right)

It can be seen in figure 10 that in the case of linear advection of the sine wave in periodic domain, the solution provided using second order ENO interpolation is the most damping one, while the interpolation using WENO combination gives slightly better results then the quadratic interpolator, both being burdened with a phase error.

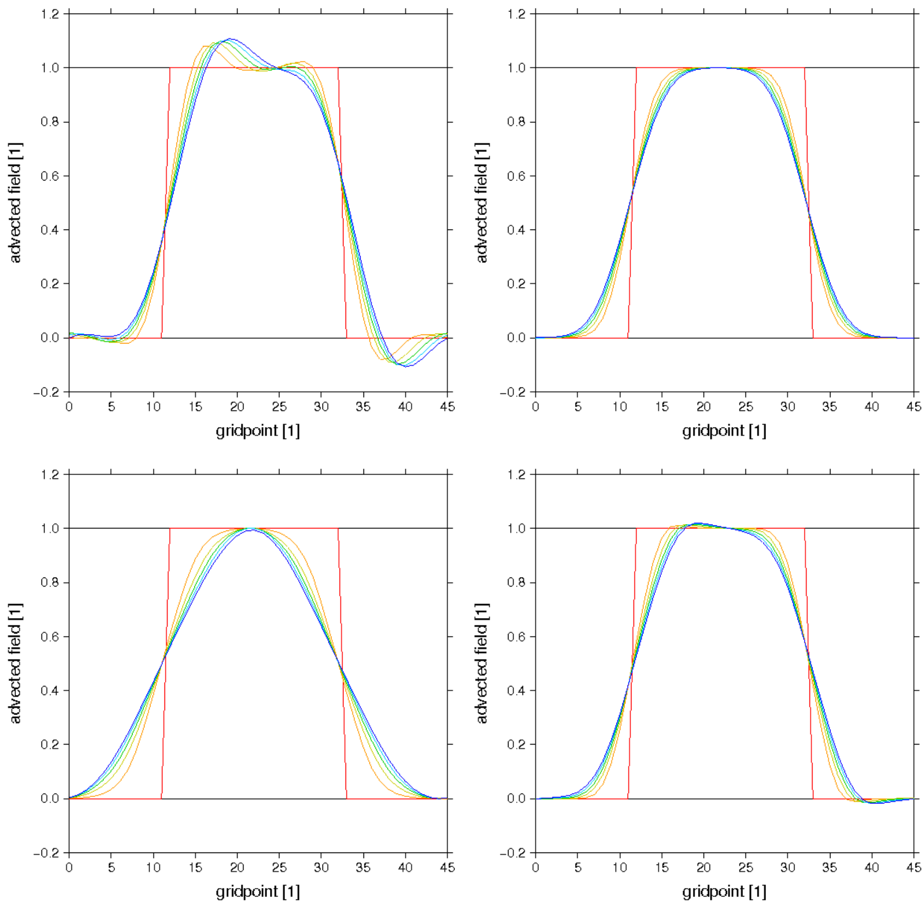


Figure 11: Linear advection of rectangular pulse in periodic domain (CFL number = 0.2): quadratic interpolator (top left), quadratic with ENO stencil (bottom left), cubic Lagrange/linear variation (top right) and quadratic with WENO combination (bottom right)

For a rectangular pulse, we get similar results concerning diffusivity and phase error, while all three proposed techniques remove overshooting present in the reference solution well, biggest residua being present in the solution with WENO. We can easily observe that while the interpolator using ENO stencil is the most damping one, the quadratic interpolator produces the biggest phase error of the four interpolators, otherwise being very similar to the quadratic with WENO combination. We can see from the figures above that the results obtained in the one-dimensional case are consistent with the ones for the two-dimensional case. On the other hand, because of the nature of this test experiments with another CFL number could give slightly different results, but other values were tested (not shown) and the solution is robust.

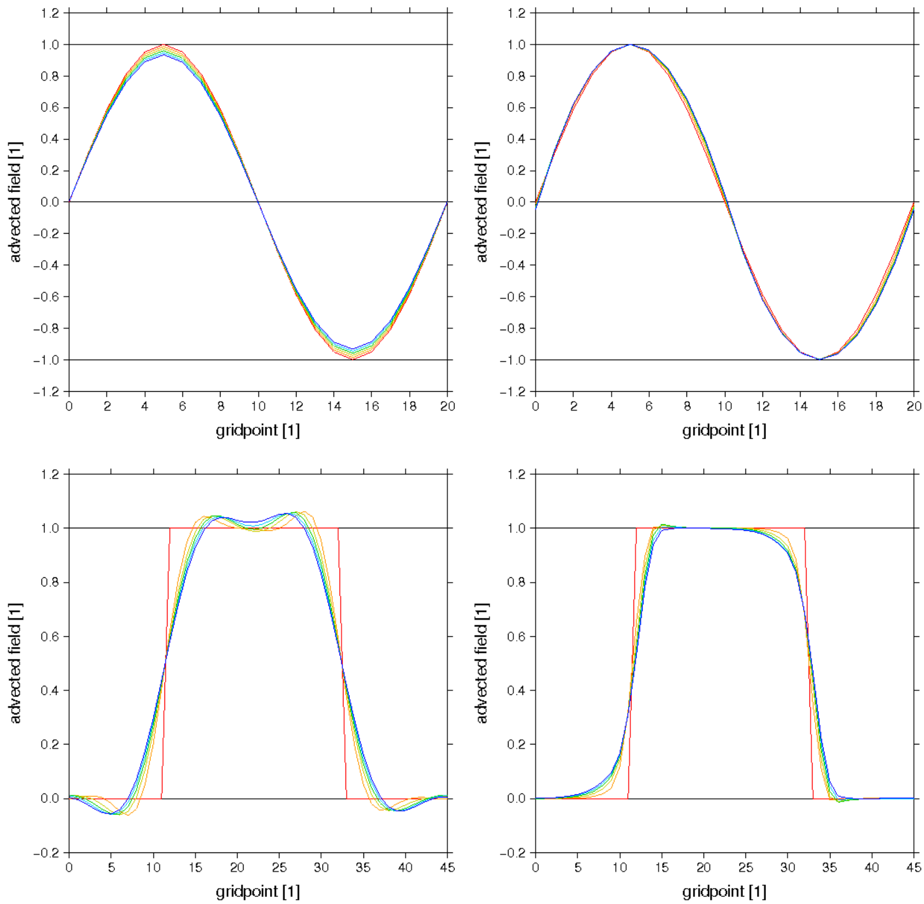


Figure 12: Linear advection of sine wave in periodic domain - first row; linear advection of rectangular pulse in periodic domain - second row: cubic Lagrange interpolator - left, cubic Lagrange and ENO stencil - right

Since the quadratic interpolators do not give satisfactory accuracy, our aim would be to apply similar techniques on at least cubic interpolators. For a 1D cubic interpolation, one needs 4 grid points and with 6 consecutive grid points 1, 2, 3, 4, 5, 6, the ENO technique will choose the smoothest solution from those obtained on $1-2-3-4$, $2-3-4-5$ and $3-4-5-6$

stencils. Thus to realize ENO or WENO methods in 1 dimension 6 points are needed. To implement cubic ENO or WENO methods in 3 dimensions, a 6x6x6 points stencil is needed in SL scheme. This makes the implementation of cubic ENO or WENO method technically demanding, and the increase in the number of high order (cubic) interpolations will create a triple increase in the computational time exigency. Those disadvantages could be balanced with the advantages of accuracy obtained by its application. This is demonstrated in 1D linear advection test.
Figure 12 shows that the third order accuracy ENO scheme (using cubic Lagrange interpolator) is the best candidate from the ones seen so far. The solution provided by this one is the most similar to the original function and is clearly better then using just cubic Lagrange interpolator. In the case of rectangular pulse, using this scheme helps to reduce the overshootings, while for the sine wave the solution provided is almost identical to the original function.

3 Conclusion

It can be concluded from the tests made that the second order ENO interpolation produces a very smooth solution such that it would not be a better replacement for the interpolators used at the moment. On the other hand, it was shown that the weighted variant, the WENO scheme could lead to better results. Another advantage of this scheme would be the higher accuracy.
Even if cubic Lagrange interpolator with ENO stencil was not yet implemented in 2D model, results obtained with 1D linear advection equation demonstrate its clear superiority over other tested approaches. One can thus hope that it will provide sharp bubble solution practically free from overshoots, more resistant against deformations caused by quasi-monotonic treatment or constant background advection. Taking this into account we consider this technique promising and worth to be tested in 2D or 3D experiments despite of being technically and computationally demanding. This work is left for future research.

Acknowledgements: I would like to thank Petra Smolíková and Ján Mašek for their help and support during the stay, as well to the whole team of CHMI for their hospitality.