Report of stay in CHMI, Prague

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These notes are a summary of the work realized during my stay in CHMI to work in the topic of the vertical discretization of the model. Splines have been introduced [UH] in the global hydrostatic model in order to increase the accuracy of the vertical scheme as basis functions of vertical finite elements. The analog in non-hydrostatic model is not feasible in a direct way by the fact that the $C1$ constraint is not satisfied. A recent developmet by Jozef Vivoda (SHMU) and Petra Smolikova (CHMI) allow overcome this by using B-splines as vertical basis functions and by an iterative method in which $\mathcal{A}_1 \neq 0$ where a set of $2L \times 2L$ equations is solved. My job was focused on the properties of B-splines, the topics where i was working on during this stay are then shown

- **Debug of the NH-VFE code:**

I had run the model changing the parameter NVFE_ORDER with several values. (which means that the degree of the associated polynomials are NVFE_ORDER-1). The experiments crashed with odd numbers. The matrix ZGP2VFE in subroutine `define_vfe_operator.F90` transforms coordinates in grid point to coefficients of splines, This matrix has sign oscillating values at non diagonal elements A_{ij} when we fix i and move j. It was observed that this matrix has different behaviuor depending on the order of splines. With a even value of NVFE_ORDER the non diagonal values oscillate with decreasing amplitude with j but not for odd values where the amplitude was increasing.

A possible explanation of the disagreement between odd and even experiments can be the fact that in even values of spline order the maxima of splines are the same as the knots only for splines with knots only with multiplicity 1, i.e., away from boundaries.

- **Search for maxima of splines:**

By a proposal of Jozef Vivoda I have developed a subroutine to compute the maximum of splines given the values of knots. Taking the values of full-leves in the maxima of splines have nice properties due to the fact that the projection operators remark a diagonal feature and reduce the values of overlaping of splines given in full-levels.

To do it we compute the values where the derivatives of splines are zero which is equivalent to find the zeroes of

$$\frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

I had computed the maximum of a single spline with the iterative Newton method taking the first guess value as $t^o := \frac{1}{k+1}(t_i + ... + t_{i+k})$. An example of a computation in given in Fig. 1

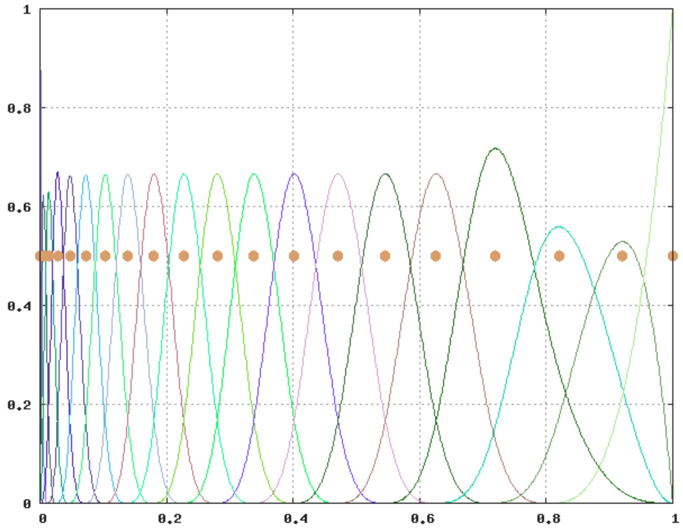

Figura 1: splines with their maxima values

- **Calculation to reshape layers:**

The spline behaviuor near the boundaries is worse than in the interior of the interval $[0,1]$. In order to arrange it I had developed a morphing function that transforms regular values of level onto a new distribution that is denser in some areas and sparse in another. To do this we can make use of a function (diffeomorphism) from the interval $[0,1]$ to $[0,1]$ that leaves the boundaries invariant

$$\begin{array}{ccc} [0,1] & \xrightarrow{f} & [0,1] \\ 0 & \mapsto & 0 \\ 1 & \mapsto & 1 \end{array}$$

The derivatives of f controls the denser or sparser areas. A function of this kind is build in a parametric way with $r \in [-1,1]$ being 0 the identity map, 1 a function that compress levels near the boundaries and -1 a function that compress in the interior of the interval. An example of this is given in Figs. 2, 3 and 4

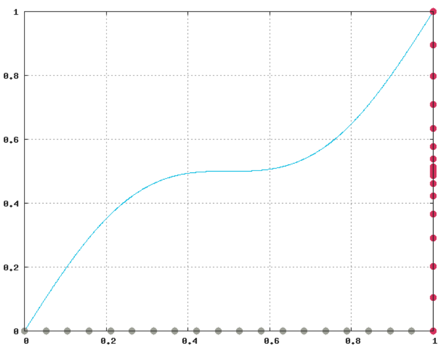

Fig. 2: r=-1

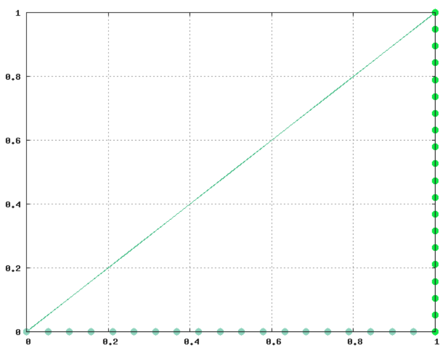

Fig. 3: r=0

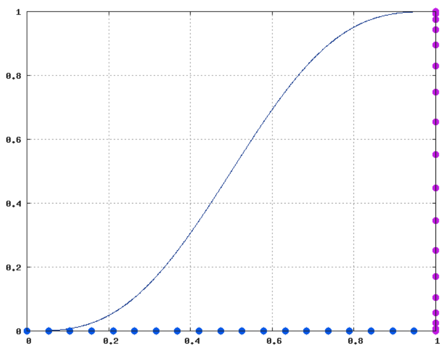

Fig. 4: r=1

- **Study of an alternative to current code:**

A nice property of B-splines is that their derivatives of a given order is a linear combination of B-splines of lower order [dB]. This is a very powerful fact that allows to express in a matricial way the derivative operator acting on splines

$$\boxed{\frac{\partial}{\partial t} N_{ik}(t) = \frac{-(k-1)}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t) + \frac{k-1}{t_{i+k-1} - t_i} N_{i,k-1}(t)} \tag{1}$$

It is possible also check that the integral operator $I := \int_{\infty}^{t} dt$ links splines of a given order with splines of increased order. It leads to an easy matricial expression with the choice of multiplicity knots at boundaries given by Jozef Vivoda. It also possible to construct a $J := \int_0^t dt$ in a similar way as I

$$\boxed{I\left[N_{ik}\right] = \frac{t_i - t_{i+k}}{k} \sum_{s=i_{\min}}^{i-1} N_{s,k+1}} \tag{2}$$

These results allow very easy computations of integral and derivatives in spline space, the associated matrices should be combined with projection operator $\mathcal{P}_k$ built as in the usual way in order to build integral and derivative operators in grid point space

$$\begin{aligned} \hat{\partial} &= \mathcal{P}_k^{-1}\, \partial\, \mathcal{P}_{k-1} \\ \hat{I} &= \mathcal{P}_k^{-1}\, I\, \mathcal{P}_{k+1} \end{aligned}$$

where ∂ and I are the associated matrices of the representation given in (1) and (2), these matrices plays the role of the product $\mathcal{A}^{-1}\mathcal{B}$ in [UH] (the inverse of mass matrix and stiff matrix). I have made an offline code to compute integral and derivative operators in this way on spline space and also on grid point space.

- **Testing off-line code:**

I have done computations with the offline code in order to calculate the integrals and derivative of a known function

$$\begin{aligned} f(x) &= \sin(\pi x)^3 \cos(\pi x) \\ \int_0^x ds f(s) &= \tfrac{1}{4\pi} \sin(\pi x)^4 \\ \tfrac{\partial}{\partial x} f(x) &= \pi[3\sin(\pi x)^2 - 4\sin(\pi x)^4] \end{aligned}$$

I have iterated calculations to compute MAE and RMSE scores with order varying from 2 to 10 and with levels from 10 to 200 with step 10. We see in the figures that the scores are better when using more levels but after a given number the results tends to stabilize. It is also shown that increasing the order of splines we get better scores, being the even values more suitable probably as it was discussed before.

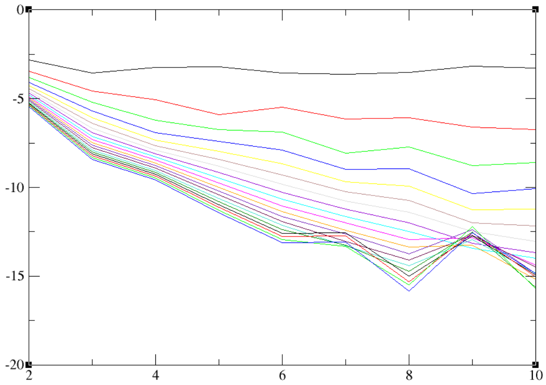

Fig. 5: $\texttt{log}_{10}\texttt{(RMSE) for } I$

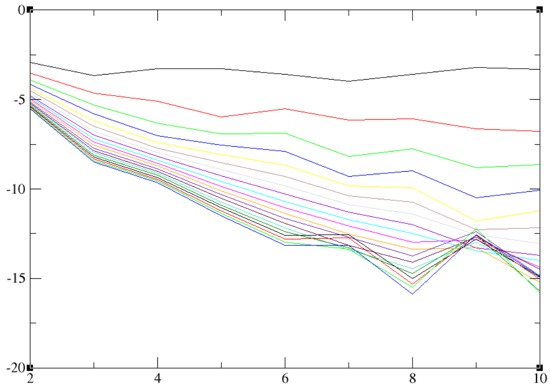

Fig. 6: $\texttt{log}_{10}\texttt{(MAE) for } I$

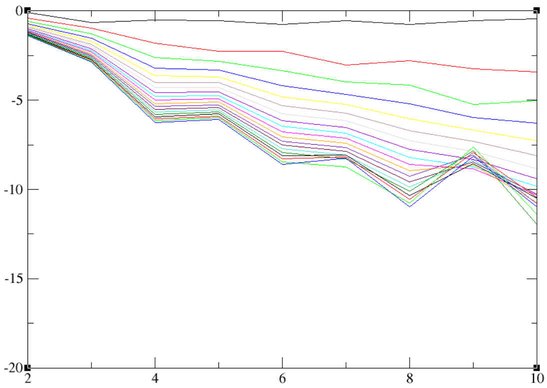

Fig. 7: $\texttt{log}_{10}\texttt{(RMSE) for } \partial$

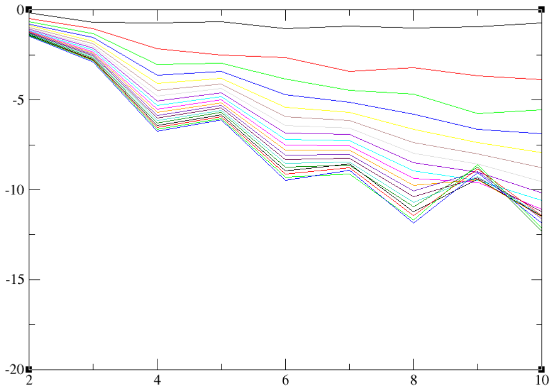

Fig. 8: $\texttt{log}_{10}\texttt{(MAE) for } \partial$

- **Implementing off-line code in model:**

Finally I have introduced the offline subroutines into the model in order to test them. The variable RINTE which is not used in nh-vfe scheme was chosen for this purpose, the integral calculations are done using this variable for spline order equal to 4. The result doesn't work at 10 levels but succeed on 110 levels with similar results of w, θ, v and T to the computations with Jozef Vivoda subroutines, the reason for this is that projection operators are no smoothed at boundaries but keep the property that the derivative and integral operators are inverse up to a constant. An implication that the operators are not smoothed at boundaries is that for few levels the computation of B_L is not accurate and the normalization $\frac{\Delta B_l}{B_L}$ done in `suvert.F90` leads to a disagreement beteween full-levels and half-levels in the sense that $\eta_l > \eta_{\tilde{l}}$. These computations were also tested on several NVFE_ORDER choices which lead to similar results as in Jozef Vivoda code, the model only works at even values. In both calculations an inverse of a projection matrix is calculated, this inverse depends on the way in which levels and knots are chosen. For even NVFE_ORDER the internal knots are levels by B-spline symmetry of construction as it is coded in `knot.F90` subroutine, but this is not true in odd case. The consequence is that the projection operator from spline to grid point space is not diagonal dominant and their inverse has an oscillatory behaviour in non diagonal-terms with increaing amplitude away from the diagonal what very probably leads to a crash of the model

Referencias

[dB] de Boor C., 1972: *On Calculating with β-Splines*. Journal of Approximation Theory 6, 50-62

[M] Marsden M., 1970: *An identity for spline functions and its application to variation dimishing spline approximations*. Journal of Approximation Theory 3, 7-49

[UH] Untch A., Hortal M., 2004: *A Finite-element Scheme for the Vertical Discretization in the Semi-langrangian Version of the ECMWF Forecast Model*. Q. J. R. Meteorol. Soc. 130, pp. 1505-1530