Extension of TOUCANS towards higher order solutions

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TOUCANS

- T - Third
- O - Order moments (TOMs)
- U - Unified
- C - Condensation
- A - Accounting and
- N - N-dependent
- S - Solver (for turbulence and diffusion)

Reynolds-averaged basic equations:

$$\frac{D\overline{u}}{\partial t} = S_u \boxed{-\frac{\partial\overline{u'w'}}{\partial z}}, \quad \frac{D\overline{v}}{\partial t} = S_v \boxed{-\frac{\partial\overline{v'w'}}{\partial z}},$$

$$\frac{D\overline{s_{sL}}}{\partial t} = S_{s_{sL}} \boxed{-\frac{\partial\overline{s'_{sL}w'}}{\partial z}}, \quad \frac{D\overline{q_t}}{\partial t} = S_{q_t} \boxed{-\frac{\partial\overline{q'_t w'}}{\partial z}}$$

u, v, w -wind components, $s_{sL} = c_{pd}\left(1 + \left[\frac{c_{pv}}{c_{pd}} - 1\right] q_t\right) T + g\,z - (L_v\, q_l + L_s\, q_i)$ a diffused moist conservative variable, g gravitational acceleration, z height, c_{pd} and c_{pv} specific heat values for dry air and water vapour, L_v and L_s latent heats of vaporisation and sublimation, T temperature, q_t total specific water content, q_l and q_i specific contents for liquid and solid water, S_ψ - external source terms, t - time, $\frac{D()}{\partial t} = \frac{\partial()}{\partial t} + \overline{u}\frac{\partial()}{\partial x} + \overline{v}\frac{\partial()}{\partial y}$, $\overline{()}$ - average, $()'$ - fluctuation

Heat and moisture flux equations

$$\frac{\partial \overline{w' s'_{sL}}}{\partial t} + \frac{\partial \overline{w'^2 s'_{sL}}}{\partial z} = + \frac{2\, O_\lambda}{C_4} \left[E_{s_{sL}} \overline{s'^2_{sL}} + E_{q_t, s_{sL}} \overline{q'_t s'_{sL}} \right] - \overline{w'^2} \frac{\partial s_{sL}}{\partial z} - \lambda_5 \frac{\overline{w' s'_{sL}}}{\tau_k}$$

$$\frac{\partial \overline{w' q'_t}}{\partial t} + \frac{\partial \overline{w'^2 q'_t}}{\partial z} = + \frac{2\, O_\lambda}{C_4} \left[E_{q_t, s_{sL}} \overline{q'^2_t} + E_{s_{sL}} \overline{s'_{sL} q'_t} \right] - \overline{w'^2} \frac{\partial q_t}{\partial z} - \lambda_5 \frac{\overline{w' q'_t}}{\tau_k}$$

tendency terms | TOMs terms | TPE contribution | cross terms | anisotropy and dissipation

O_λ - free parameter, λ_5, C_4 - coefficients, N^2 - Brunt–Väisälä frequency , $E_{s_{sL}}(SCC)$, $E_{q_t, s_{sL}}(SCC)$ - buoyancy weights according to (Marquet and Geleyn, 2013), SCC - Shallow Convection Cloudiness, τ_k - dissipation time scale

Variance equations

$$\frac{\partial \overline{s_{sL}'^2}}{\partial t} + \frac{\partial \overline{w' s_{sL}'^2}}{\partial z} = -2 \frac{\partial s_{sL}}{\partial z} \overline{w' s_{sL}'} - \frac{4\, C_3}{C_4} \frac{\overline{s_{sL}'^2}}{\tau_k}$$

$$\frac{\partial \overline{q_t'^2}}{\partial t} + \frac{\partial \overline{w' q_t'^2}}{\partial z} = -2 \frac{\partial q_t}{\partial z} \overline{w' q_t'} - \frac{4\, C_3}{C_4} \frac{\overline{q_t'^2}}{\tau_k}$$

equilibrium

$$\frac{\partial \overline{w'^2}}{\partial t} + \frac{\partial \overline{w'^3}}{\partial z} = \text{Source terms} - \frac{2}{\lambda} \frac{\overline{w'^2}}{\tau_k}$$

equilibrium

neglected TOMs

e_k - TKE, $I = -\overline{u' w'} \frac{\partial u}{\partial z} - \overline{v' w'} \frac{\partial v}{\partial z}$ - shear term in TKE equation

Anisotropy and dissipation contribution

- assuming equilibrium condition
- neglecting all terms except dissipation and anisotropy contribution

$$\overline{w's'_{sL}} = -\frac{\tau_k \overline{w'^2}}{\lambda_5}\frac{\partial s_{sL}}{\partial z} = -K'_H \frac{\partial s_{sL}}{\partial z}$$

$$\overline{w'q'_t} = -\frac{\tau_k \overline{w'^2}}{\lambda_5}\frac{\partial q_t}{\partial z} = -K'_H \frac{\partial q_t}{\partial z}$$

$$K'_H = C_3 \frac{\nu^4}{C_\epsilon} L \sqrt{e_k^+}\,\phi_Q(Ri_f)$$

K'_H - anisotropy exchange coefficient, ϕ_Q - anisotropy stability dependency function, L - length scale, Ri_f - flux Richardson number, ν and C_ϵ - free parameters, C_3 - inverse Prandtl number at neutrality

Prognostic Total Turbulent Energy (TTE)

- parametrisation of counter-gradient heat transport maintained by velocity shear following (Zilitinkevich et al., 2013)
- pair of prognostic turbulent energies - TKE and TTE
- equilibrium assumption links energy ratio share $\Pi = \frac{TTE - TKE}{TKE}$ to stability parameters Ri_f, Ri
- Π used as new stability parameter
- usage of TKE solver also for TTE

Turbulent energy equations in moist case

$$e_t = e_p^* + e_k$$

$$\frac{De_k}{dt} + \text{TOM term} = +I + II - \frac{2\, e_k}{\tau_k}$$

$$\frac{De_p^*}{dt} + \text{TOM term} = -II - \frac{2\, e_p^*}{\tau_p}$$

$$\frac{De_t}{dt} + \text{TOM term} = +I - \frac{2\, e_t}{\tau_t}$$

equilibrium

expected shape according to dry case

$$I = -\overline{u'w'}\frac{\partial u}{\partial z} - \overline{v'w'}\frac{\partial v}{\partial z}, \quad II = E_{s_{sL}}\, \overline{w's'_{sL}} + E_{q_t, s_{sL}}\, \overline{w'q'_t}$$

τ_p or τ_t needs to be determined

e_k - Turbulent Kinetic Energy (TKE), e_p^* - reservoir of Turbulent Potential Energy (TKE), e_t- Total Turbulent Energy (TTE), I - shear term, II - buoyancy term, τ_p and τ_t - dissipation time scales

Reservoir of TPE

- linear combination of variance equations for heat and moisture, which give expected TPE equation shape:

$$\frac{De_p^*}{dt} + \mathrm{TOM\ term} = -E_{s_{sL}}\,\overline{w's'_{sL}} - E_{q_t,s_{sL}}\,\overline{w'q'_t}$$

$$-\frac{E_{s_{sL}}}{2\frac{\partial s_{sL}}{\partial z}}\frac{4\,C_3\,\overline{s'^2_{sL}}}{C_4\,\tau_k} - \frac{E_{q_t,s_{sL}}}{2\frac{\partial q_t}{\partial z}}\frac{4\,C_3\,\overline{q'^2_t}}{C_4\,\tau_k}$$

$$\frac{De_p^*}{dt} + \mathrm{TOM\ term} = -II - \frac{4\,C_3\,e_p^*}{C_4\tau_k}$$

$$e_p^* = \frac{E_{s_{sL}}\,\overline{s'_{sL}2}}{2\frac{\partial s_{sL}}{\partial z}} + \frac{E_{q_t,s_{sL}}\,\overline{q'^2_t}}{2\frac{\partial q_t}{\partial z}},\ \tau_p = \frac{C_4\tau_k}{4\,C_3\,e_p^*}$$

Equilibrium condition

- expression for τ_t:

$$I + II = \frac{2\, e_k}{\tau_k}, \quad -II = \frac{4\, C_3\, e_p^*}{C_4 \tau_k}, \quad I = \frac{2\, e_t}{\tau_t} \Rightarrow$$

$$\tau_t = \frac{e_t}{\frac{e_k}{\tau_k} + \frac{e_p^*}{\frac{C_4}{2\, C_3}\tau_k}} = \tau_k \frac{C_4\,(1+\Pi)}{C_4 + 2\, C_3 \Pi}, \quad \Pi \equiv \frac{e_p^*}{e_k} = \frac{e_t}{e_k} - 1$$

- relation to flux Richardson number at equilibrium:

$$Ri_f \equiv \frac{-II}{I} = \frac{\Pi}{\frac{C_4}{2\, C_3} + \Pi}$$

Heat-Moisture covariance equation

$$\frac{\partial \overline{q_t' s_{sL}'}}{\partial t} + \mathrm{TOMs\ term} \quad = \quad -\frac{\partial s_{sL}}{\partial z}\overline{w' q_t'} - \frac{\partial q_t}{\partial z}\overline{w' s_{sL}'} - \frac{4\ C_3\ \overline{q_t' s_{sL}'}}{C_4\ \tau_k}$$

we assume equilibrium condition and get:

$$\frac{4\ C_3\ \overline{q_t' s_{sL}'}}{C_4\ \tau_k} \quad = \quad -\frac{\partial s_{sL}}{\partial z}\overline{w' q_t'} - \frac{\partial q_t}{\partial z}\overline{w' s_{sL}'}$$

TPE contribution at equilibrium

- assuming equilibrium condition
- neglecting tendency terms and TOMs terms

$$\frac{4\,C_3}{C_4}\frac{\overline{s_{sL}'^2}}{\tau_k} = -2\frac{\partial s_{sL}}{\partial z}\overline{w's_{sL}'}, \quad \frac{4\,C_3}{C_4}\frac{\overline{q_t'^2}}{\tau_k} = -2\frac{\partial q_t}{\partial z}\overline{w'q_t'}$$

$$\overline{w's_{sL}'} = -K_H'\frac{\partial s_{sL}}{\partial z} - K_H'\frac{O_\lambda\,\tau_k}{2\,C_3\overline{w'^2}}\cdot \left(2\,E_{s_{sL}}\overline{w's_{sL}'}\frac{\partial s_{sL}}{\partial z} + E_{q_t,s_{sL}}\left[\frac{\partial s_{sL}}{\partial z}\overline{w'q_t'} + \frac{\partial q_t}{\partial z}\overline{w's_{sL}'}\right]\right)$$

$$\overline{w'q_t'} = -K_H'\frac{\partial q_t}{\partial z} - K_H'\frac{O_\lambda\,\tau_k}{2\,C_3\overline{w'^2}}\cdot \left(E_{s_{sL}}\left[\frac{\partial s_{sL}}{\partial z}\overline{w'q_t'} + \frac{\partial q_t}{\partial z}\overline{w's_{sL}'}\right] + 2\,E_{q_t,s_{sL}}\overline{w'q_t'}\frac{\partial q_t}{\partial z}\right)$$

TPE contribution at equilibrium

$$\overline{w' s'_{sL}} = -K'_H \frac{\partial s_{sL}}{\partial z} - K'_H \frac{O_\lambda \tau_k}{2\, C_3 \overline{w'^2}} \left(\overline{w' s'_{sL}}\, N^2 + \frac{\partial s_{sL}}{\partial z}\, II \right)$$

$$\overline{w' q'_t} = -K'_H \frac{\partial q_t}{\partial z} - K'_H \frac{O_\lambda \tau_k}{2\, C_3 \overline{w'^2}} \left(\overline{w' q'_t}\, N^2 + \frac{\partial q_t}{\partial z}\, II \right)$$

$$\overline{w' s'_{sL}} \left(1 + K'_H \frac{O_\lambda \tau_k}{2\, C_3 \overline{w'^2}}\, N^2 \right) = -K'_H \frac{\partial s_{sL}}{\partial z} \left(1 + \frac{O_\lambda \tau_k}{2\, C_3 \overline{w'^2}}\, II \right)$$

$$\overline{w' q'_t} \left(1 + K'_H \frac{O_\lambda \tau_k}{2\, C_3 \overline{w'^2}} N^2 \right) = -K'_H \frac{\partial q_t}{\partial z} \left(1 + \frac{O_\lambda \tau_k}{2\, C_3 \overline{w'^2}}\, II \right)$$

TPE contribution at equilibrium

$$N^2 = \frac{8\,\Pi}{C_4\,\frac{K_H C_\epsilon}{C_3\,L\sqrt{e_k}}\tau_k^2}, \quad II = -\frac{4\,C_3\,\Pi\,e_k}{C_4 \tau_k}$$

equilibrium

$$\overline{w' s'_{sL}} = -K_H \frac{\partial s_{sL}}{\partial z}, \quad \overline{w' q'_t} = -K_H \frac{\partial q_t}{\partial z}$$

$$K_H = K'_H \left(1 - \frac{O_\lambda}{C_4\,\overline{w'^2}}\,\Pi\right)$$

$$= C_3\,\frac{\nu^4}{C_\epsilon} L\sqrt{e_k}\,\phi_Q(Ri_f)\left(1 - \frac{O_\lambda}{C_4\,\overline{w'^2}}\,\Pi\right)$$

anisotropy energy conversion

$$= C_3\,\frac{\nu^4}{C_\epsilon} L\sqrt{e_k}\,\phi_3(\Pi) \longleftarrow$$ TPE contribution via Π

ϕ_3 - stability dependency function for heat and moisture

TPE contribution - prognostic

- both TKE and TTE can be treated prognostically - usage of the same solver
- the link between energy ratio - Π and Ri_f is kept as in equilibrium condition
- shape of ϕ_3 stability dependency function is kept as in equilibrium condition

Third Order Moments (TOMs) contribution

- distant turbulent transport caused by presence of semi-organised large eddies
- parametrisation for heat and moisture
- following (Canuto, Cheng, and Howard, 2007):

$$\overline{w'\theta'} = -K_H \frac{\partial \overline{\theta}}{\partial z} + A_1^\theta \frac{\partial \overline{w'^3}}{\partial z} + A_2^\theta \frac{\partial \overline{w'\theta'^2}}{\partial z} + A_3^\theta \frac{\partial \overline{w'^2\theta}}{\partial z}$$

$$\overline{w'^3} = -0.06 \frac{g}{\theta} \tau_k^2 \overline{w'^2} \frac{\partial \overline{w'\theta'}}{\partial z}, \quad \overline{w'\theta'^2} = -\tau_k \overline{w'\theta'} \frac{\partial \overline{w'\theta'}}{\partial z}, \quad \overline{w'^2\theta'} = -0.3 \tau_k \overline{w'^2} \frac{\partial \overline{w'\theta'}}{\partial z}$$

θ - potential temperature, A_1^θ, A_2^θ, A_3^θ - coefficients

TOMs contribution - moist case

$$\overline{w's'_{sL}} = -K_H \frac{\partial s_{sL}}{\partial z} + A_1^{s_{sL}} \frac{\partial \overline{w'^3}}{\partial z} + A_2^{s_{sL}} \frac{\partial \overline{w's'^2_{sL}}}{\partial z} + A_3 \frac{\partial \overline{w'^2 s'_{sL}}}{\partial z},$$

$$\overline{w'q'_t} = -K_H \frac{\partial q_t}{\partial z} + A_1^{q_t} \frac{\partial \overline{w'^3}}{\partial z} + A_2^{q_t} \frac{\partial \overline{w'q'^2_t}}{\partial z} + A_3 \frac{\partial \overline{w'^2 q'_t}}{\partial z},$$

$$\overline{w's'^2_{sL}} = -\tau_k \overline{w's'_{sL}} \frac{\partial \overline{w's'_{sL}}}{\partial z}, \qquad \overline{w'q'^2_t} = -\tau_k \overline{w'q'_t} \frac{\partial \overline{w'q'_t}}{\partial z}$$

$$\overline{w'^2 s'_{sL}} = -0.3\tau_k \overline{w'^2} \frac{\partial \overline{w's'_{sL}}}{\partial z}, \qquad \overline{w'^2 q'_t} = -0.3\tau_k \overline{w'^2} \frac{\partial \overline{w'q'_t}}{\partial z}$$

$$\overline{w'^3} = -0.06\tau_k^2 \overline{w'^2} \left(E_{s_{sL}} \frac{\partial \overline{w's'_{sL}}}{\partial z} + E_{q_t, s_{sL}} \frac{\partial \overline{w'q'_t}}{\partial z} \right)$$

$A_{1-3}^{s_{sL}/q_t}$ - coefficients

TOMs contribution - moist case

- from equation for variances and heat and moisture fluxes:

$$
\begin{aligned}
A_1^{s_{sL}} &= K^{(A_1)}\, K_H\, T_h \frac{\mathcal{T}_k}{e_k} \frac{\partial s_{sL}}{\partial z}, \quad A_1^{q_t} = K^{(A_1)}\, K_H\, T_h \frac{\mathcal{T}_k}{e_k} \frac{\partial q_t}{\partial z} \\
A_2^{s_{sL}} &= -K^{(A_2)} E_{s_{sL}} K_H\, T_h \frac{\mathcal{T}_k}{e_k}, \quad A_2^{q_t} = -K^{(A_2)} E_{q_t, s_{sL}} K_H\, T_h \frac{\mathcal{T}_k}{e_k} \\
A_3 &= -K^{(A_3)}\, K_H\, T_h \frac{1}{e_k}, \\
T_h &= \frac{1}{\overline{w'^2}} \frac{2\, \phi_Q}{\phi_Q + \phi_3}, \\
K^{(A_1)} &= \frac{\lambda}{2}, \quad K^{(A_2)} = \frac{O_\lambda}{2\, C_3}, \quad K^{(A_3)} = 1.
\end{aligned}
$$

TOMs contribution - two step solver

- local diffusion:

$$\frac{\partial s_{sL}^{\mathrm{loc}}}{\partial t} = \frac{\partial\left(-g\rho K_H \frac{\partial s_{sL}^{\mathrm{loc}}}{\partial z}\right)}{\partial p}$$

$$\frac{\partial q_t^{\mathrm{loc}}}{\partial t} = \frac{\partial\left(-g\rho K_H \frac{\partial q_t^{\mathrm{loc}}}{\partial z}\right)}{\partial p}$$

- TOM s contribution
  - updates of these references with increments $\delta s_{sL}^{+} = s_{sL}^{+} - s_{sL}^{\mathrm{loc}}$ and $\delta q_t^{+} = q_t^{+} - q_t^{\mathrm{loc}}$
  - stable and accurate algorithm immune against singularities
  - requires iterations to improve accuracy

Tendency contribution

$$\overline{w' s'_{sL}} + A_t \frac{\partial \overline{w' s'_{sL}}}{\partial t} = -K_H \frac{\partial s_{sL}}{\partial z} + A_1^{s_{sL}} \frac{\partial \overline{w'^3}}{\partial z} + A_2^{s_{sL}} \frac{\partial \overline{w' s'^2_{sL}}}{\partial z} + A_3 \frac{\partial \overline{w'^2 s'_{sL}}}{\partial z},$$

$$\overline{w' q'_t} + A_t \frac{\partial \overline{w' q'_t}}{\partial t} = -K_H \frac{\partial q_t}{\partial z} + A_1^{q_t} \frac{\partial \overline{w'^3}}{\partial z} + A_2^{q_t} \frac{\partial \overline{w' q'^2_t}}{\partial z} + A_3 \frac{\partial \overline{w'^2 q'_t}}{\partial z}$$

$$A_t = -K^{(A_3)} K_H T_h \frac{1}{e_k}$$

- parametrisation via scaling parameter by assuming $\psi^{-} - \psi^{--} \cong \psi^{\mathrm{loc}} - \psi^{-}$

TOMs contribution - solver

$$\frac{\delta \mathbf{s}_{\mathbf{sL}}^{+\,[i+1]}}{\delta t} =$$

$$\frac{1}{1+\frac{A_t}{\delta t}}\Bigg[\frac{\partial\left(\left[-g\rho K_H - g\rho K_H \frac{T_h T_{**}^{s_{sL}}}{\delta t}\right]\frac{\partial\left(\delta \mathbf{s}_{\mathbf{sL}}^{+\,[i+1]}\right)}{\partial z}\right)}{\partial p} + \frac{\partial\left(\rho K_H . T_h(\{T_*^{-1}\}^{s_{sL}}\widehat{\left(\frac{\delta \mathbf{s}_{\mathbf{sL}}^{+\,[i+1]}}{\delta t}\right)})\right)}{\partial p}$$

$$+\frac{\partial\left(-g\rho K_H\left(\frac{T_h T_{**}^{s_{sL}}}{\delta t}\frac{\partial\left(s_{sL}^{\mathrm{loc}}-s_{sL}^{-}\right)}{\partial z}\right)\right)}{\partial p} + \frac{\partial\left(\rho K_H . T_h(\{T_*^{-1}\}^{s_{sL}}\widehat{\left(\frac{s_{sL}^{\mathrm{loc}}-s_{sL}^{-}}{\delta t}\right)})\right)}{\partial p}\Bigg]$$

$$+\frac{\partial\left(-g\rho K_H\left(\frac{T_h T_{cr}^{sq}}{\delta t}\frac{\partial\left(\overbrace{K_{cr}^{sq}\ e_k\left[q_t^{[i]}-q_t^{-}\right]}\right)}{\partial z}\right)\right)}{\partial p}$$

The $\overbrace{\quad}$ operator is used for interpolating from half levels to full levels and the $\widehat{\quad}$ operator for interpolating from full levels to half levels, '$^{+}$' and '$^{-}$' marking next and current time steps, '$^{[i]}$' marking value from the i-th iteration of the TOMs solver except for $i = 1$ where it marks values computed from the local diffusion solver

TOMs contribution - solver

$$
\begin{aligned}
\{T_*^{-1}\}^{s_{sL}} &= g(K^{(A_1)}\frac{6}{100}\overbrace{\frac{\partial s_{sL}}{\partial z}(\frac{\tau_k}{\widehat{e}_k})}\frac{\partial\left(E_{s_{sL}}\,2\,A_z\tau_k^2\widehat{e}_k\right)}{\partial z} \\
&\quad -K^{(A_3)}\frac{3}{10}(\frac{1}{e_k})\frac{\partial\,(2\,A_z\tau_k\widehat{e}_k)}{\partial z} - K^{(A_2)}\overbrace{E_{s_{sL}}(\frac{\tau_k}{\widehat{e}_k})}\frac{\partial\left(\tau_k\left(\overline{w's'_{sL}}\right)^{[i]}\right)}{\partial z}) \\
T_{**}^{s_{sL}} &= -K^{(A_1)}\frac{6}{100}E_{s_{sL}}\frac{\partial s_{sL}}{\partial z}2\,A_z\tau_k^3 + K^{(A_3)}\frac{3}{10}2\,A_z\tau_k + K^{(A_2)}\frac{\tau_k^2}{\widehat{e}_k}E_{s_{sL}}\left(\overline{w's'_{sL}}\right)^{[i]}, \\
T_{cr}^{sq} &= -K^{(A_1)}\frac{6}{100}\frac{\partial s_{sL}}{\partial z}\frac{\tau_k}{\widehat{e}_k}, \quad K_{cr}^{sq} = E_{q_t,s_{sL}}\tau_k^2 2\,A_z
\end{aligned}
$$

Heat and moisture flux influenced by:

- anisitropy and dissipation term
- TPE contribution:
  - equilibrium
  - prognostic
- TOMs contribution
  - heat and moisture with separate non-local transport
  - with cross terms due to $\overline{w'^3}$
  - with iteration
- tendency term parametrisation

Thank you for your attention!