ALARO TRAINING COURSE

GOVERNING EQUATIONS (L02)

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WARNING

This presentation is an updated and extended version of the presentation shown at the TCWGPDI in Prague (22-26 November 2004)

Equationwise there are no big differences but we have now a more sound physical reasoning and an extension towards the compressible case

The full details of the first part is given in Catry *et al.* (2007), Tellus 59A, 71-79

OUTLINE OF THIS LESSON

Part 1:
physical-mathematical development of the set of governing equations

Part 2:
where in the code do we find these equations ?

WHICH EQUATIONS ?

The governing equations are those which control the interaction between the physics and dynamics of a model (i.e. the Physics-Dynamics-Interface)

In particular we shall develop here in an as general as possible way the equations which describe the conservation of the different prognostic species and the corresponding thermodynamical equation to update the temperature

WHY A NEW SET OF EQUATIONS ?

The diabatic part of most atmospheric models seldom has a set of governing equations built on the basis of sound scientific principles, because:

• "plug-compatibility", i.e. the belief that exchangeability was more important than consistency

• some parameterisation packages were developed under constraints only imposed by the dynamical core of their particular host model

• parameterisations and discretisations evolved too fast for a successful attempt to find consistent equations

not so valid anymore

SIMPLIFYING HYPOTHESES

- The atmosphere is in permanent thermodynamic equilibrium

- Condensed phases are assumed to have a zero volume

- All gases (dry air and water vapour) obey Boyle-Mariotte's and Dalton's law

- All specific heat values are independent of temperature

- The temperature of all species inside an elementary portion are assumed to be the same one

The Species

Consider a micro-physical scheme including the following 6 mass species:

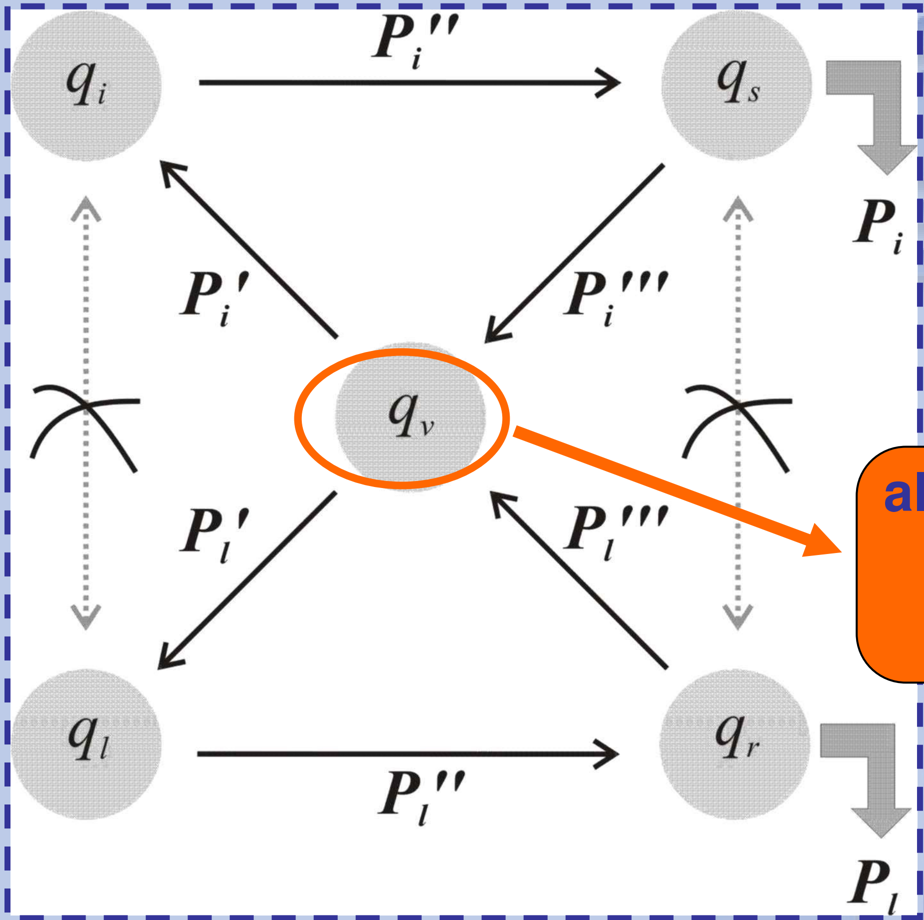
1. dry air (q_d)
2. water vapor (q_v)
3. suspended liquid water (q_l)
4. rain water (q_r)
5. suspended ice (q_i)
6. snow (q_s)

$$\boxed{q_x = \frac{\rho_x}{\rho}}$$

no hail or graupel
(thermodynamically the same as snow)

THE SCHEME

All phase changes can be redefined by the following scheme



all phase changes go through the vapor phase

A MASS WEIGHTED SYSTEM

The equations will be developed in a mass weighted (barycentric) system

⬇

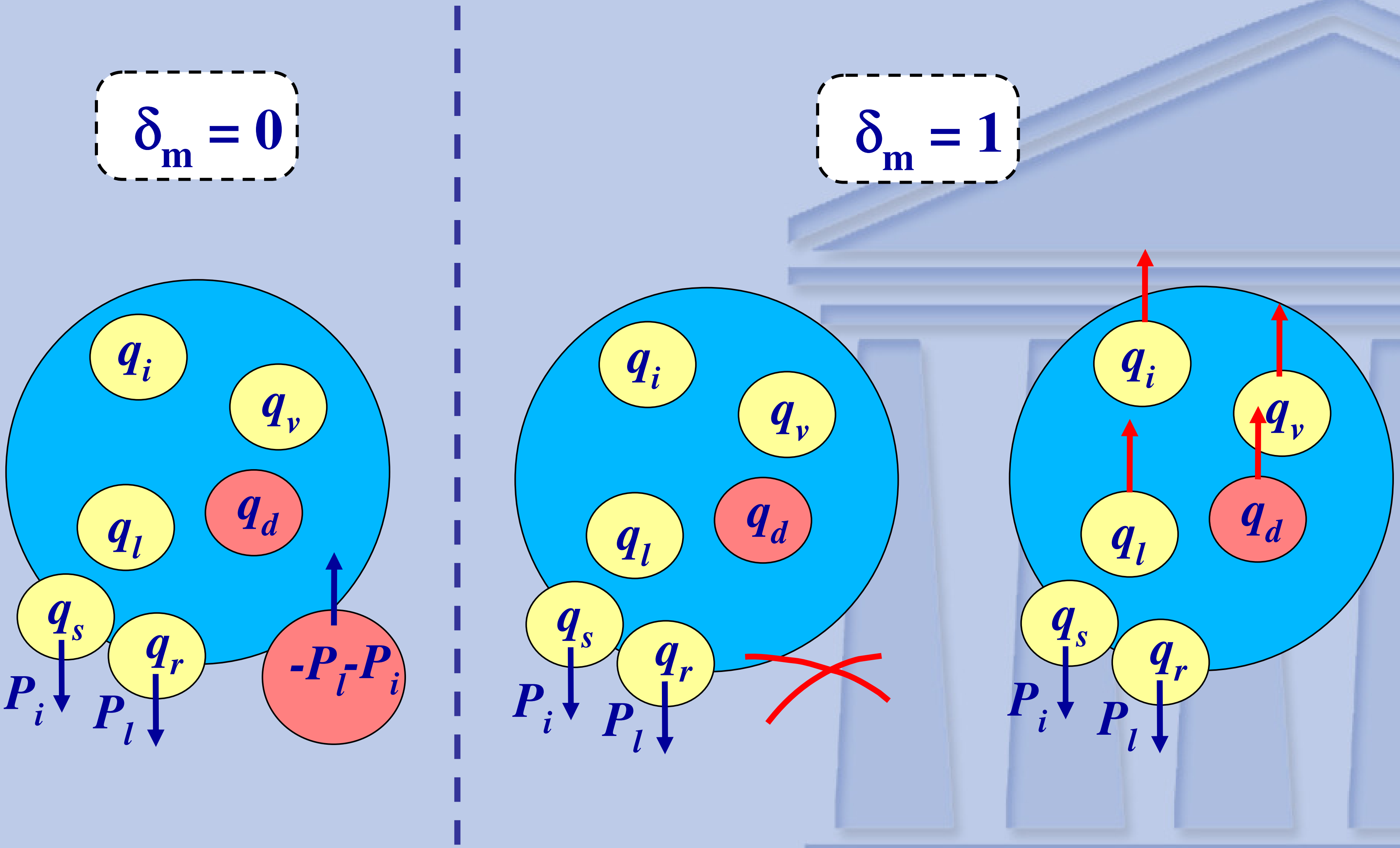
Velocities and fluxes are written with respect to the centre of mass

⬇

e.g. $P_l = \rho_r w_r$ and $P_i = \rho_s w_s$

All non-precipitating species are assumed to have the same vertical velocity

A MASS WEIGHTED SYSTEM (2)



ADDITIONAL CONSTRAINT

We shall consider from now on

$$\boxed{\delta_m = 1}$$

(loss of mass due to precipitation
is NOT compensated with flux of dry air)

Mass Weighted Asymmetry

When using a mass weighted framework the following asymmetry arises:

rain drops and snow flakes cannot compress during their descent

$\updownarrow$

dry air and water vapour experience a compensating lift and these species can then expand

Mass Weighted Asymmetry (2)

This asymmetry implies the use of two frameworks:

1. a fixed one which works with the gaseous species only
2. a full mass weighted one which treats all species

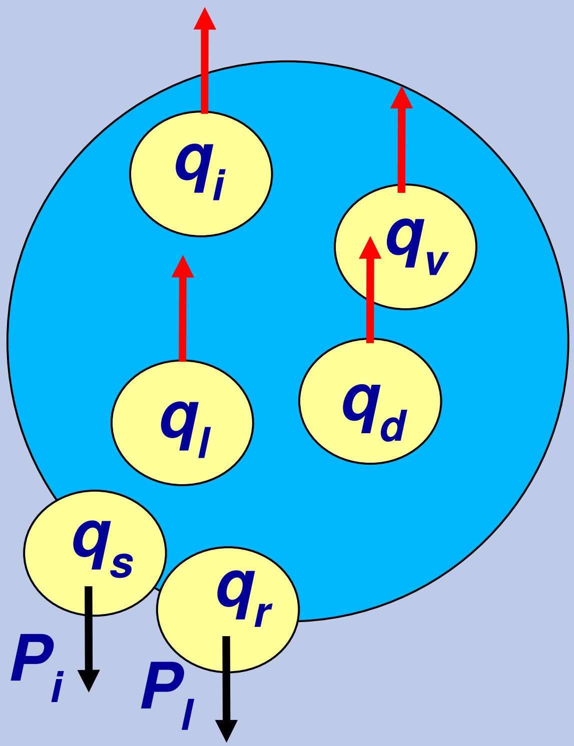
The transition between these two frameworks:

$$\left.\frac{d}{dt}\right|_{all} = \left.\frac{d}{dt}\right|_{gas} + g(P_l + P_i)^* \frac{\partial}{\partial p}$$

or

$$\omega_{all} = \omega_{gas} + g(P_l + P_i)^*$$

MASS WEIGHTED FLUXES

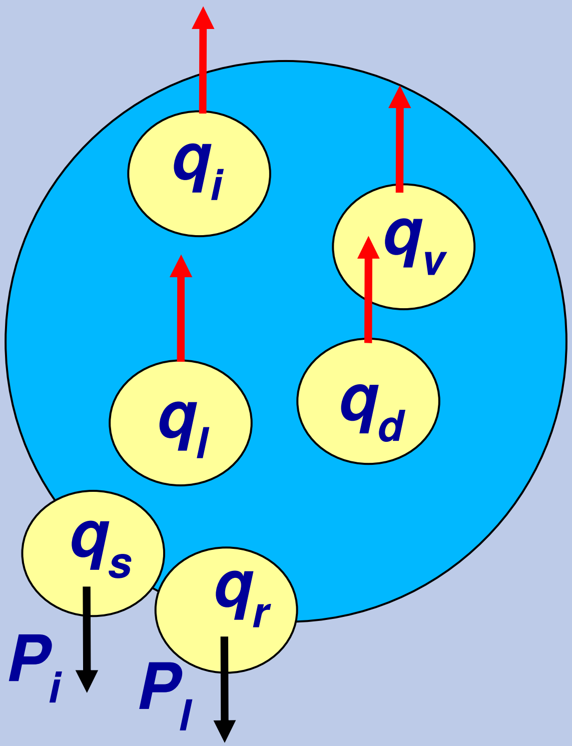


In the barycentric environment, the fluxes of the non-precipitating species should cancel out the total precipitation flux $P_l + P_i$

Assuming that all non-precipitating species move with the same speed with respect of the centre of mass, the compensating flux for e.g. the vapor part can be written as:

$$\frac{q_v(P_l + P_i)}{q_a + q_v + q_l + q_i} = \frac{q_v(P_l + P_i)}{1 - q_r - q_s}$$

Mass weighted velocities



From the fluxes with respect to the centre of mass we can compute the relative velocities of the non-precipitating species:

$$P_n = \rho_n w_n = -\frac{q_n (P_l + P_i)}{1 - q_r - q_s} = -\frac{\rho_n (P_l + P_i)}{\rho (1 - q_r - q_s)}$$

$$w_n = -\frac{P_l + P_i}{\rho (1 - q_r - q_s)}$$

CONTINUITY EQUATION + CO

In the barycentric case there are no mass fluxes acting as source terms

continuity equation →

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \eta}\right) = -\nabla.\left(\vec{v}\frac{\partial p}{\partial \eta}\right) - \frac{\partial}{\partial \eta}\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)$$

surface pressure →

$$\frac{\partial \pi_s}{\partial t} = -\int_0^1 \nabla.\left(\vec{v}\frac{\partial p}{\partial \eta}\right) d\eta - g(E+R+S)$$

vertical velocity →

$$\omega = \vec{v}.\nabla p - \int_0^\eta \nabla.\left(\vec{v}\frac{\partial p}{\partial \eta}\right) d\eta$$

independent of the precipitation flux!

CONSERVATION OF SPECIES

The conservation of water vapour can be written as:

$$\frac{\partial}{\partial t}\left(q_v\frac{\partial p}{\partial \eta}\right) = -\nabla.\left(q_v\vec{v}\frac{\partial p}{\partial \eta}\right) - \frac{\partial}{\partial \eta}\left(q_v\dot{\eta}\frac{\partial p}{\partial \eta}\right)$$

$$-g\frac{\partial}{\partial \eta}\left(P_l' - P_l''' + P_i' - P_i''' - \frac{q_v(P_l + P_i)}{1 - q_r - q_s}\right)$$

$$\frac{\partial q_v}{\partial t} = -g\frac{\partial}{\partial p}(P_l' + P_i') + g\frac{\partial}{\partial p}(P_l''' + P_i''') + g\frac{\partial}{\partial p}\left(q_v\frac{P_l + P_i}{1 - q_r - q_s}\right)$$

'additional flux'

CONSERVATION OF SPECIES (2)

$$
\begin{aligned}
\frac{\partial q_v}{\partial t} &= -g\frac{\partial}{\partial p}(P_l' + P_i') + g\frac{\partial}{\partial p}(P_l''' + P_i''') + g\frac{\partial}{\partial p}\left(q_v\frac{P_l + P_i}{1 - q_r - q_s}\right) \\
\frac{\partial q_l}{\partial t} &= +g\frac{\partial P_l'}{\partial p} - g\frac{\partial P_l''}{\partial p} + g\frac{\partial}{\partial p}\left(q_l\frac{P_l + P_i}{1 - q_r - q_s}\right) \\
\frac{\partial q_r}{\partial t} &= +g\frac{\partial P_l''}{\partial p} - g\frac{\partial P_l'''}{\partial p} - g\frac{\partial P_l}{\partial p} \\
\frac{\partial q_i}{\partial t} &= +g\frac{\partial P_i'}{\partial p} - g\frac{\partial P_i''}{\partial p} + g\frac{\partial}{\partial p}\left(q_i\frac{P_l + P_i}{1 - q_r - q_s}\right) \\
\frac{\partial q_s}{\partial t} &= +g\frac{\partial P_i''}{\partial p} - g\frac{\partial P_i'''}{\partial p} - g\frac{\partial P_i}{\partial p} \\
\frac{\partial q_a}{\partial t} &= +g\frac{\partial}{\partial p}\left(q_a\frac{P_l + P_i}{1 - q_r - q_s}\right)
\end{aligned}
$$

THERMODYNAMIC EQUATION

The full thermodynamic equation can be expressed as

$$c_p \frac{dT}{dt} - RT \frac{d \ln p}{dt} + g \frac{1}{\rho} (P_l + P_i)^* = Q$$

with R the gas constant, Q the diabatic heat source and $(P_l+P_i)^*$ the absolute precipitation flux (non-barycentric)

The additional term (with respect to the general form of the thermodynamic equation) represents the dynamical impact of the formation or evaporation of precipitation

SOME REMARKS

$$c_p \frac{dT}{dt} - RT\frac{d\ln p}{dt} + g\frac{1}{\rho}(P_l + P_i)^* = Q$$

This expression for the thermodynamic equation was derived without any reference to whether one uses HPE or compressible conditions

The additional term expresses the dynamical impact of the formation or evaporation of precipitation

This additional term only appears when the acceleration due to precipitation is not countered by something

Two cases → hydrostatic case
→ compressible case

Hydrostatic case

The hydrostatic approximation filters out the adjustment waves caused by the formed or evaporated precipitation

$$c_p \frac{dT}{dt} - RT \frac{d \ln p}{dt} = Q'$$

(with Q' the diabatic heat source out of which the dynamical impact is filtered)

The physical tendency of enthalpy becomes

$$\frac{\partial}{\partial t}(c_p T) = -g \frac{\partial}{\partial p} [(c_l - c_{pd}) P_l T + (c_i - c_{pd}) P_i T - (\hat{c} - c_{pd})(P_l + P_i) T + J_s + J_{rad}$$
$$- L_l(T_0)(P_l' - P_l''') - L_i(T_0)(P_i' - P_i''')]$$

where we used $L_{l|i}(T) = L_{l|i}(T_0) + (c_{pv} - c_{l|i})T$

and where $\hat{c} = \dfrac{c_{pd} q_a + c_{pv} q_v + c_l q_l + c_i q_i}{1 - q_r - q_s}$

HYDROSTATIC CASE (2)

We finally have the following flux-conservative equation

$$\boxed{\frac{d(c_pT)}{dt} - RT\frac{d\ln p}{dt} = -g\frac{\partial J_{total}}{\partial p}}$$

COMPRESSIBLE CASE

remember

$$c_p \frac{dT}{dt} - RT\frac{d\ln p}{dt} + g\frac{1}{\rho}(P_l + P_i)^* = Q$$

and

$$\omega_{all} = \omega_{gas} + g(P_l + P_i)^*$$

$$c_p \frac{dT}{dt} - \frac{p_{gas}}{\rho}\frac{d\ln p_{gas}}{dt} = Q$$

with

$$p_{gas} = \rho_{gas} R_{gas} T$$

$$\rho_{gas} = \rho_d + \rho_v$$

$$R_{gas} = \frac{1}{\rho_d + \rho_v}(\rho_d R_d + \rho_v R_v) = \frac{\rho}{\rho_{gas}} R$$

COMPRESSIBLE CASE (2)

$$c_p \frac{dT}{dt} - \frac{p_{gas}}{\rho}\frac{d\ln p_{gas}}{dt} = Q$$

using

$$\frac{p_{gas}}{\rho} = \frac{\rho_{gas} R_{gas} T}{\rho} = RT$$

$$c_v \frac{dT}{dt} + RTD_3 = Q + T\frac{dR}{dt}$$

When doing the inverse computation, one finds back

$$c_p \frac{dT}{dt} - RT\frac{d\ln p}{dt} + g\frac{1}{\rho}(P_l + P_i)^* = Q$$

This tells us that identifying the pressure field by the pressure of gases filters out the impact of the vertically moving species

PROJECTION OF HEAT SOURCE

We finally find the following set of equations for both temperature and pressure changes

$$
\begin{aligned}
c_v \frac{dT}{dt} + RT\ D_3 &= \\
c_v \frac{d\ln(p)}{dt} + c_p D_3 &=
\end{aligned}
$$

$$
\begin{aligned}
\frac{d(c_v T)}{dt} + RT\ D_3 &= -g \frac{\partial J_{total}}{\partial \pi} \\
\frac{d\ln(c_v p/R)}{dt} + \frac{c_p}{c_v} D_3 &= -\frac{g}{c_v T} \frac{\partial J_{total}}{\partial \pi}
\end{aligned}
$$

CASE $\delta_m = 0$

We then have the following flux-conservative equation

$$\frac{\partial}{\partial t}(c_p T) = -g\frac{\partial}{\partial p}\Big[(c_l - c_{pd})P_l T + (c_i - c_{pd})P_i T - \cancel{(\hat{c} - c_{pd})(P_l + P_i)T} + J_s + J_{rad}$$

$$-L_l(T_0)(P_l' - P_l''') - L_i(T_0)(P_i' - P_i''')\Big]$$

SUMMARY

We have the following flux-conservative equation (for both the hydrostatic and compressible case):

$$\frac{d(c_pT)}{dt} - RT\frac{d\ln p}{dt} = -g\frac{\partial J_{total}}{\partial p}$$

$$\frac{\partial}{\partial t}(c_pT) = -g\frac{\partial}{\partial p}\left[(c_l - c_{pd})P_lT + (c_i - c_{pd})P_iT - (\hat{c} - c_{pd})(P_l + P_i)T + J_s + J_{rad}\right.$$
$$\left. - L_l(T_0)(P_l' - P_l''') - L_i(T_0)(P_i' - P_i''')\right] = -g\frac{\partial J_{total}}{\partial p}$$

Compressible case: projection of heat source on both temperature and pressure changes:

$$\begin{aligned} c_v\frac{dT}{dt} + RT\ D_3 &= Q' + T\frac{dR}{dt} \\ c_v\frac{d\ln(p)}{dt} + c_pD_3 &= \frac{Q'}{T} + \frac{c_p}{R}\frac{dR}{dt} \end{aligned}$$

$$\begin{aligned} \frac{d(c_vT)}{dt} + RT\ D_3 &= -g\frac{\partial J_{total}}{\partial \pi} \\ \frac{d\ln(c_vp/R)}{dt} + \frac{c_p}{c_v}D_3 &= -\frac{g}{c_vT}\frac{\partial J_{total}}{\partial \pi} \end{aligned}$$

OUTLINE OF THIS LESSON (BIS)

Part 1:
physical-mathematical development of the set of governing equations

Part 2:
where in the code do we find these equations ?

DATA FLOW: APLPAR

APLPAR

calls to parameterisations

↓

all kinds of FLUXES

(TURB, CONV, STRAT, DIFF)

↓

CPTEND_NEW

DATA FLOW: CPTEND_NEW

CPTEND_NEW

computation of physical tendencies

switch between $\delta_m=0$ and $\delta_m=1$

no need for CPFHPFS

PTENDH, PTENDU, PTENDV, PTENDQ, PTENDQL, PTENDQI, PTENDQR, PTENDQS

no tendency of pressure yet

CPUTQY

DATA FLOW: CPUTQY

CPUTQY

computation of physical increments

↓

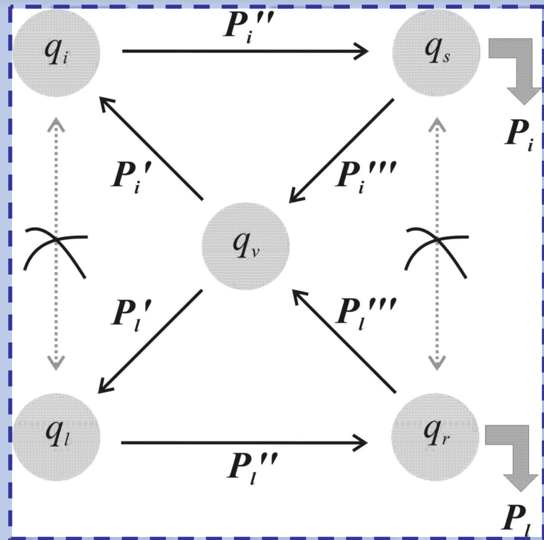
update of T, u, v, q_v , q_l , q_i , q_r and q_s

no update of pressure yet

IMPORTANT !!

When developing parameterisations with some impact on the prognostic species, obey the ideas behind these governing equations:

- the output should be in flux-form (no tendencies)
- these fluxes should be defined following the simplified scheme of (pseudo-) fluxes



- for now there is only one switch inside CPTEND_NEW: choice of δ_m
- future: projection of heat source on pressure change

QUESTIONS ??

Time for questions / remarks / comments