APLMPHYS and its ingredients

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A) What existed in ACPLUIE:

- A mechanism of condensation and evaporation aimed to return at the 'wet bulb point' $[T_w, q_w]$. The choice of this target (rather than $[T, q_{sat}]$) allows hiding the thermodynamics in the interfacing. In case of super-saturation, the full equilibrium is found in one time step (no suspended water, infinite fall-speed of precipitations having a zero volume). In case of under-saturation, reaching the equilibrium fixes the maximum rate that the evaporation of precipitation can take.
- A rule for the evaporation of falling rain/snow. Given the above, it is entirely thought in term of fluxes (zero x infinity can give a bounded value and indeed does here) and ignores any precipitating species' quantity. The computation depends on the under-saturation level of the encountered layer both for its upper limit (see above) and its intensity.

The formulation is as follows (explanations in Appendix 1):

$$\frac{d\sqrt{R}}{d(1/p)} = E_{vap}.(q_w - q)$$

with $\boldsymbol{R}$ the rainfall rate (in kg/m**2/s) and $\boldsymbol{p}$ the pressure (in Pa). The default numerical value of the constant (EVAP in the code) is 4.8 10^6.

- A parallel rule for the melting/freezing of the falling precipitations. The same remark as for the evaporation applies. The computation now depends on the difference between the local temperature and the treble point one. Here also the thermodynamics is hidden in the change of type of precipitating fluxes, at equal sum for both phases.

The formulation is as follows (explanations in Appendix 2):

$$\frac{d\,m_i}{d(1/p)} = F_{ont}.(T - T^*)/\sqrt{R}$$

with $\boldsymbol{m_i}$ the thermodynamical snow proportion of the precipitation flux and $\boldsymbol{T^*}$ the treble point temperature (in °K). The default numerical value of the constant (FONT in the code) is 2.4 10^4.

- A distinction between the mechanical and thermodynamic properties of the mixed case when both precipitating phases exist together. The proportion of condensed ice depends on the temperature below the treble point, this giving continuity for the function that reaches zero above the treble point. Then the melting/freezing rule is applied, but there is never an 'average latent heat'. On the contrary the proportionality constants for the melting/freezing and evaporation rates are higher for more slowly falling precipitations and are hence modulated by the square root of the ratio 'fall speed of rain over fall speed of snow' (REVGSL in the code).

Hence we have symbolically:

$$E_{vap} = EVAP\sqrt{(1-m_e.(1-REVGSL))} \qquad \& \qquad F_{ont} = FONT\sqrt{(1-m_e.(1-REVGSL))}$$

with $\boldsymbol{m_e}$ ($\neq \boldsymbol{m_i}$) the physical proportion of ice in the precipitation flux. This differential treatment is a kind of simple substitute for accounting graupel. In the case of a prognostic microphysical scheme we do not have the facility of such a clean distinction, but some diagnostic-type accounting of a pseudo-graupel flux and/or phase is necessary.

The points corresponding to the three first bullets were carried through to APLMPHYS, keeping an algorithmic formulation as close as possible to the one of ACPLUIE. The adaptation steps are rather obvious and will not be described in this mini-documentation. The fourth one (apart from the analytical formulation of E_{vap} and F_{ont}) will be very different but shall rely on computations similar to those for water and pure snow. This is described in Appendix 8.

As last necessary issue to get closer in realism to hydrometeors' behaviour in nature, the APLMPHYS code has been extended to take into account, contrary to ACPLUIE, geometrical considerations. This complex step is fortunately made easier by the central use of statistical (or PDF-based) sedimentation algorithms. A brief description of the employed methods is given in Appendix 9, while the exact mechanism can best be understood by looking directly at the code.

In extension of what exists in ACPLUIE, the link with the thermodynamical equation is ensured outside APLMPHYS through the reliance on the equations of Catry et al. (2007).

According to the spirit of the 3MT backbone for ALARO-0, there is no description in this note about the condensation processes leading to the appearance of q_l and q_i, since the 'other' microphysical computations should be treated independently of the origin and magnitude of the condensation/re-evaporation fluxes controlling the budgets of q_l and q_i.

B) <u>The sedimentation problem</u>:

- When going to a solution where there is a prognostic treatment of both cloud water ***[q_l, q_i]*** and precipitating species ***[q_r, q_s]***, it seems very difficult to imagine an algorithm preserving the existing characteristics of ACPLUIE and just adding new ones, if the sedimentation of falling precipitating species is treated advectively. Indeed neither the time iterations of an Eulerian treatment nor the double vertical loops of a Lagrangian one match well the simplicity and compactness of the single vertical loop of ACPLUIE, something we aim to preserve in order to create a fully prognostic version of a truly Kessler-type scheme.
- The problem comes from the fact that advective methods require a unique (mean) fall speed for any type of precipitation. Let us however assume that this is replaced by a spectrum of fall velocities going from zero to infinity, the

above-mentioned unique fall speed still being the mass weighted average of this spectrum, i.e. the first moment of the associated PDF.

- For a given time interval (typically the model's time step) the PDF may easily be converted in one of 'reachable distances along the vertical'. At that stage one is apparently in an even worse situation than previously, with an infinity of trajectory to handle.
- But if one assumes that the PDF is a decreasing exponential $\boldsymbol{P_0}$, one can use the fundamental property (similar to the one of constant life expectancy of one radio-active isotope during its disintegration) that the normalised probabilities are identical whatever the origin of the drops leaving a model layer at its bottom:
    i. Already present in the layer at the beginning of the time step (PDF $\boldsymbol{P_1}$);
    ii. Coming from the layer above (PDF $\boldsymbol{P_2}$);
    iii. Locally produced by auto-conversion, collection or melting (PDF $\boldsymbol{P_3}$).
- Hence one can linearly combine the three 'sources' in question to get a unique PDF at the bottom of the layer and thus at the top of the next one below, this allowing to repeat the same computations everywhere within a single vertical loop. Indeed, in the next layer below, one will not need any information on the origin of the precipitations to be handled as point 'ii' of the process, while of course undergoing the 'local' processes (melting-freezing, evaporation, collection, …). One hereby replaces the cumbersome advective treatment of sedimentation by a simple statistical one.
- If one assumes an infinite average fall speed, the method immediately degenerates in the one used for the non-prognostic version of ACPLUIE. This helps explaining why this choice, apart from other advantages, was the best one for the specific goal of constructing APLMPHYS.
- The proposed PDF of fall velocities and the Dirac-type alternative for the advective methods (that may in fact also be mathematically transformed in $\boldsymbol{P_{1/2/3}}$ functions) are both quite far away from the observed truth but the former is probably a bit more realistic. There is even some similarity with the Marshall-Palmer law, when assuming a variable $\boldsymbol{N_0}$ and a linear dependency of fall speed upon size (rather acceptable assumptions) as well as a volume of drops independent of their fall speed (rather wrong of course). Hence the link is surely true qualitatively but not quantitatively. The difference is due to the mass-weighted choice with respect to the size-weighted one of the observed law.
- Since there is no more risk to get numerical problems with such an enhancement, one may introduce an observed-like dependency of the average fall speed upon the intensity of the precipitation flux. Strictly speaking this is not compatible with the additivity rule central to the proposal, but since one would have to deal with rather small variations of the parameter across each model layer, it is empirically numerically acceptable (and more realistic of course).

Mathematically speaking one gets (explanations in Appendix 3):

$$P_0(\delta z/(\overline{w}.\delta t)) = P_0(Z) = e^{-Z}$$
$$P_1(Z) = (1 - P_0(Z))/Z$$
$$P_3(Z) = (1/2 - E_3(Z))/Z$$
$$P_2(Z) = (E_2(Z) + P_3(Z))/(1 + P_3(Z))$$

with $\boldsymbol{E_{2/3}}$ the second and third exponential integrals, that we shall need to approximate.

C) <u>The parameterisation of new processes</u>:

- ***Fall speed dependency on the intensity of the flux***; for water (the extension to snow will be detailed later):

$$\overline{w} = 13.4\left(\frac{R}{\rho^4}\right)^{1/6} \qquad \text{with } \rho \text{ the air density (in kg/m**3)}$$

Details can be found in Appendix 4. The obtained mean fall speed enters the '$\boldsymbol{Z}$' computation for the statistical sedimentation. In order to avoid a perpetuation of zero (or too weak) values for this fall speed, an estimate of $\boldsymbol{R}$ at the middle of the model layer is used, assuming full condensation and conversion. The constant (of recommended value 13.4 SI units) is named $\boldsymbol{\Omega^r}$.

- ***Collection***; one maxes the hypothesis that the cloud water is continuously present and hence collected along the volume scanned by the falling rain drops, with a tunable efficiency factor (0.2) multiplying the result; for water (the extension to snow and/or ice cases will also be detailed later):

$$\frac{dq_l}{dt} = -0.335.E_{ff}.R^{4/5}.q_l = -0.067.R^{4/5}.q_l$$

Details can be found in Appendix 5. The numerical treatment uses a split-implicit algorithm with respect to $\boldsymbol{q_l}$. Alike in the previous bullet, an estimate of $\boldsymbol{R}$ (relying this time on the more precise results of the auto-conversion computations, see below) is used to compute the time scale of the $\boldsymbol{q_l}$ depletion. The constant (of recommended value 0.067 SI units) is named $\boldsymbol{C'^r_E}$.

- ***Auto-conversion***; in the spirit of the old ACPLUIE very basic scheme, one uses the most simple and continuous possible expression (Sundquist type).

$$\left(\frac{dq_{l/i}}{dt}\right)_{ACO} = -\frac{q_{l/i}}{\tau_{l/i}(T)}\left(1 - e^{-\frac{\pi}{4}(q_{l/i}/q^{cr}_{l/i}(T))^2}\right)$$

The characteristic times and critical 'threshold' values for water and ice (with a temperature dependency in the second case) will be among the main tuning parameters of the scheme. The ***π/4*** factor is there to scale the Sundquist-type formula at the same level as the Kessler-type one.

The ***Wegener-Bergeron-Findeisen (WBF) process*** is parameterised as an auto-conversion from cloud water to snow. The formula will be similar to the ones for 'classical' auto-conversion.

$$\left(\frac{dq_l}{dt}\right)_{WBF} = -F_{WBF}^{a}\,\frac{q_l}{\tau_l}\,\frac{q_l.q_i}{(q_l+q_i)^2}\left(1-e^{-\frac{\pi}{4}((q_l.q_i)/(q_l^{cr}.F_{WBF}^{b}.q_i^{cr}(T).F_{WBF}^{b})}\right)$$

The two additional WBF scaling constants will also be important tuning parameters of the scheme.

Details can be found in Appendix 6. For all three terms (liquid to rain, ice to snow and liquid to snow), the numerical treatment of auto-conversion also uses a split-implicit algorithm with respect to $q_{l/i}$, while assuming explicit values for the remaining parts of the expressions. In both case (collection and auto-conversion) these split-implicit algorithms take into account the previously computed processes, as well as the combination of effects on either q_l or q_i.

D) <u>The introduction of temperature dependencies for the case of ice and/or snow</u>:

- While the mechanical and thermodynamical properties of cloud droplets and of raindrops hardly depend on the ambient temperature, there exist several such dependencies in the case of ice crystals and/or snowflakes. These dependencies are introduced on the basis of the figures given by Lopez (2002) but with some simplifications.
- In the reduced framework of APLMPHYS, there are five such dependencies to be taken into account:
  1. How does the auto-conversion time scale (for ice to snow) vary with temperature (the colder the air, the less efficient is the process)?
  2. How does the critical threshold for ice to snow auto-conversion vary with temperature between a minimum and a maximum (at ***0°C***) value (the colder the air, the lower is the threshold)?
  3. How does the fall speed of snow vary with temperature (the colder the air, the slower is the fall)?
  4. How does the collection efficiency factor for ice crystals vary with temperature (the colder the temperature, the more 'escaping' the crystals are)?
  5. How does the falling snow scanned volume for collection varies, mostly through a N_0 and a dependency, with temperature (the colder

the temperature, the more flakes with a smaller size and thus a better surface to volume ratio, hence the more efficient is the collection)?

- In Lopez (2002) all dependencies are treated through the use of functions of the type $exp(c_t(T-T^*))$ (with T^* roughly equal to the $0°C$ temperature), except the one for item (ii) above. The latter is treated with a tangent hyperbolic function but we elected to also extend the exponential formulation to this case. Using the constraint to intercept the other function at its inflexion point one gets $c_t=\alpha ln(2)/\beta$ in Lopez' notations.
- In case of item (iii), Equations B3 and B4 of Lopez (2002) would allow an immediate extension to the above-mentioned formalism in $R^{1/6}$, but for the fact that the quantity under the ***1/6*** power is the specific amount of falling species and not the precipitation flux itself (the ratio between the two is the fall speed, indeed a rather constant parameter). We elected to neglect the difference between both (weak) dependencies and to directly use a ratio of fall speeds of ***3.959*** at $0°C$ and the exponential function of equation B4.
- The case corresponding to item (v) is more complex and interacts with the hypothesis made in the previous bullet as well as with the formulations developed in Appendix 4 and Appendix 5 for the dependency of Ω^r and C^r_E upon a and N_0. See Appendix 7 for the details.
- All this taken into account, the 'basic' c_t values for each item are the following:
  - i. 0.025;
  - ii. 0.02195;
  - iii. 0.0204;
  - iv. 0.025;
  - v. 0.02365.

Given the proximity of all these values it was decided to use a single dependency with the geometrical average of all five values ($c_t^*=0.0231$) and no modification of the values at $0°C$. Among the latter, it should be stressed that the N_0 for snow is a quarter of that for rain (***2.E+06***) and that the collection efficiency of snow is half the one of rain (***0.1***).

We shall name $f_{s/i}^*(T)=exp(c_t^*(T-T^*))$.

Appendix 1: Kessler-type method for the evaporation of rain

One starts with the Marshall-Palmer distribution law of drop-sizes, i.e.

$$N = N_0\, e^{-\lambda D} \qquad\qquad N_0 = 8.\,10^6$$

that one wishes to combine with the following anticipated fit to measurements (those being taken from the Smithsonian Tables, values at 20°C and 1 atm) for the fall speed and the rate of mass depletion by evaporation, e.g.

$$w = a\,D^{\alpha} \qquad\qquad a(p,T)$$

$$\frac{dM}{dt} = b\,D^{\beta}\,\rho(q - q_w) \qquad\qquad b(p,T)$$

One then gets:

$$R = \int_0^{\infty} a\,D^{\alpha} N_0\, e^{-\lambda D} \frac{\pi D^3}{6 v_l} dD = \frac{\Gamma(\alpha+4) a N_0 \pi}{6 v_l\, \lambda^{\alpha+4}}$$

$$\frac{dR}{dz} = \int_0^{\infty} b\,D^{\beta} N_0\, e^{-\lambda D}\,\rho(q_w - q)\,dD = \rho(q_w - q)\frac{\Gamma(\beta+1) b N_0}{\lambda^{\beta+1}}$$

Eliminating λ and going to pressure thicknesses gives:

$$\frac{dR}{dp} = \frac{\Gamma(\beta+1) b N_0}{g}\left(\frac{6 v_l R}{\Gamma(\alpha+4) a N_0 \pi}\right)^{\frac{\beta+1}{\alpha+4}} (q - q_w)$$

$$\frac{dR^{1-\frac{\beta+1}{\alpha+4}}}{dp} = (1 - \frac{\beta+1}{\alpha+4})\frac{\Gamma(\beta+1) b N_0}{g}\left(\frac{6 v_l}{\Gamma(\alpha+4) a N_0 \pi}\right)^{\frac{\beta+1}{\alpha+4}} (q - q_w) = A(q - q_w)$$

Analysis of the above-mentioned data gives $\boldsymbol{\alpha}$ = 0.7706 and $\boldsymbol{\beta}$ = 1.614, this leading to an exponent for $\boldsymbol{R}$ of value 0.4521. Given the other approximations of the method, one may round it to 0.5 by modifying $\boldsymbol{\beta}$ to 1.3853 (the value of $\boldsymbol{\alpha}$ is more certain). One of course redoes the fit with this imposed condition in order to get a new function for $\boldsymbol{b}$. One obtains (always with SI units):

$$a = 654.5(\rho_0 / \rho)^{\alpha}$$

$$b = 0.005655(T_0 / T)^{7.095}(p_0 / p)$$

$$\Rightarrow \qquad A = 4.146\ 10^{-4}(T_0 / T)^{7.4803}(p_0 / p)^{0.6147}$$

One now eliminates the ***T*** dependency by going to the standard atmosphere profile as new reference:

$$p_0 = 101325. \quad T_0 = 288.16 \quad dT/dz = -0.0065 \quad =>$$

$$A = 4.678\ 10^{-4}\ (p_0/p)^{2.038} \approx 4.678\ 10^{-4}\ (p_0/p)^2 = \frac{4.802\ 10^6}{p^2} \approx \frac{4.8\ 10^6}{p^2}$$

Appendix 2: Kessler-type method for the melting of snow into rain

One uses the same ingredients than in Appendix 1 plus a formula for the depletion of the ice mass of the 'drop' during melting (that uses the ratio between the molecular diffusion of heat and the molecular diffusion of water vapour):

$$\left(\frac{dM_i}{dt}\right)_f = b' D^\beta \rho (T^* - T) \qquad b' = b \frac{\gamma}{\Delta} \frac{C_{pd}}{L_f} = b\,B \qquad B = 2.522\ 10^{-3}$$

with also

$$R_i = m_i\, R \qquad \& \qquad M_i = m_i\, M$$

one sees that

$$\frac{dR_i}{dz} = \frac{d(Rm_i)}{dz} = m_i \frac{dR}{dz} + R\frac{dm_i}{dz} = -\frac{dM_i}{dt} = -m_i \frac{dM}{dt} - \left(\frac{dM_i}{dt}\right)_f = m_i \frac{dR}{dz} - \left(\frac{dM_i}{dt}\right)_f$$

Hence, going to the whole spectrum without changing the $\boldsymbol{m_i}$ notation, one gets, for the sole melting process:

$$\frac{dm_i}{dp} = \frac{6 v_l b'}{g a \pi}(T^* - T)\frac{\int_0^\infty N_0 e^{-\lambda D} D^\beta dD}{\int_0^\infty N_0 e^{-\lambda D} D^{3+\alpha} dD} = \frac{6 v_l b' \Gamma(\beta+1)}{g a \pi \Gamma(\alpha+4)}(T^* - T)\,\lambda^{(\alpha+4)-(\beta+1)}$$

$$= \frac{6 v_l b' \Gamma(\beta+1)}{g a \pi \Gamma(\alpha+4)}\left(\frac{6 v_l R}{\Gamma(\alpha+4) a N_0 \pi}\right)^{\frac{\beta+1}{\alpha+4}-1}(T^* - T)$$

$$= \frac{AB}{1 - \dfrac{\beta+1}{\alpha+4}} R^{\frac{\beta+1}{\alpha+4}-1}(T^* - T)$$

And one finally obtains (with the simplifying choices of Appendix 1):

$$\frac{dm_i}{dp} = \frac{2AB}{\sqrt{R}}(T^* - T) \qquad A \approx \frac{4.8\ 10^6}{p^2} \qquad \& \qquad B \approx 2.5\ 10^{-3}$$

Given its symmetry, this formula will also be applied in case of rain freezing.

Appendix 3: Basic computations for the statistical sedimentation

All computations rely on the basic quantity P_0 that expresses the probability to cross one layer in one time step. It has no direct use (this is too restrictive a case) but it will be useful in all three cases of interest described below. Furthermore its basic homothetic-preservation property is at the root of the proposed 'statistical sedimentation' method, even if other functions might lead to other interesting schemes.

$$P_0(\delta z/(\overline{w}.\delta t)) = P_0(Z) = e^{-Z}$$

Case of the raindrops present in the layer at the beginning of the time-step

One assumes a homogeneous distribution in space. Hence

$$P_1(Z) = \frac{1}{\delta z}\int_0^{\delta z} P_0(z,\delta t)\,dz = \frac{1}{\delta z}\int_0^{\delta z} e^{-\frac{z}{\overline{w}.\delta t}}\,dz = \frac{\overline{w}.\delta t}{\delta z}\left[1-e^{-\frac{\delta z}{\overline{w}.\delta t}}\right] = (1-P_0(Z))/Z$$

Case of the raindrops coming from the layer above

One starts by simply assuming a homogeneous arrival in time (at the top). Hence a first expression is obtained as

$$P'_2(Z) = \frac{1}{\delta t}\int_0^{\delta t} P_0(\delta z,t)\,dt = \frac{1}{\delta t}\int_0^{\delta t} e^{-\frac{\delta z}{\overline{w}.t}}\,dt = E_2(\delta z/(\overline{w}.\delta t)) = E_2(Z)$$

This will be approximated by

$$P'_2(Z) = E_2(Z) = \frac{P_0(Z)}{Z+1+X} \qquad with \qquad X = \frac{\sqrt{(1+Z)^2+4Z}-(1+Z)}{2}$$

The link between P'_2 and the final P_2 function will be explained below, after derivation of the related P_3 function.

Case of the raindrops created in the layer during the time-step

One assumes a homogeneous production in time and space. One thus gets the convolution of the two previous computations

$$P_3(Z) = \frac{1}{\delta z}\int_0^{\delta z}\frac{1}{\delta t}\int_0^{\delta t} P_0(z,t)\,dt\,dz = \frac{1}{\delta z}\int_0^{\delta z} E_2(z/(\overline{w}.\delta t))\,dz = \left(\frac{1}{2}-E_3(Z)\right)/Z$$

This will be approximated by

$$P_3(Z)=\left(\frac{1}{2}-E_3(Z)\right)/Z=0.5(E_2(Z)+P_1(Z))=0.5(P_2'(Z)+P_1(Z))$$

when using the exact recurrence relationship between $\boldsymbol{E_3}$ and $\boldsymbol{E_2}$ ($\boldsymbol{2E_3(Z)+Z.E_2(Z)=P_0(Z)}$), thanks to the above-chosen approximation for $\boldsymbol{E_2}$ (the $\boldsymbol{X}$ term would indeed be identical if approximating $\boldsymbol{E_3}$ by $\boldsymbol{P_0(Z)/(Z+2+X)}$).

Now, the above derivation of the ***P1***, ***P'2*** and ***P3*** functions does not take into account what kind of redistribution happens between the various terms within the layer along the time step, in order to be compatible with the flux-driven balance equation that will ultimately be used to compute the evolution of falling species. One thus must look further at the problem, under a double angle: the stationary solution and the full evolution equation of $\boldsymbol{q_r}$ (or equivalently $\boldsymbol{q_s}$) if no stationarity is warranted.

<u>Stationary case</u>:

$$\frac{\partial q_r}{\partial t}\equiv 0 \quad\Rightarrow\quad P_l^{bot}-P_l^{top}=\frac{\delta p}{g.\delta t}(\Delta_q^{aco}-\Delta_q^{eva})$$

but, by definition:

$$P_l^{bot}=P_l^{top}.P2+\frac{\delta p}{g.\delta t}(\Delta_q^{aco}-\Delta_q^{eva}).P3+q_r\frac{\delta p}{g.\delta t}.P1$$

Hence we have:

$$P_l^{bot}(1-P3)=P_l^{top}(P2-P3)+q_r\frac{\delta p}{g.\delta t}.P1$$

and a sound physical solution can in principle only be obtained with $\boldsymbol{P2\geq P3}$.

<u>Evolution equation of q_r</u>. In the above basic derivation of all four functions, the ***P1*** and ***P3*** parts can be seen as intangible values resulting from the computation of:

$$\frac{1}{\delta z}\int_0^{\delta z}[P_0(z,t).q_r+\frac{1}{\delta t}\int_0^{\delta t}P_0(z,t)(\Delta_q^{aco}-\Delta_q^{eva})dt]dz$$

$$=\frac{1}{\delta z}\int_0^{\delta z}\frac{1}{\delta t}\int_0^{\delta t}P_0(z,t).\tilde{q}_r(t).dt\,dz$$

But the true evolution equation of $\boldsymbol{q_r}$ (without the tilde) also contains a term coming from the vertical divergence of the precipitation fluxes at the edges (Catry et al., 2007) and hence related to ***P2***. One may start by saying that the part of the flux at the top which will not be subject to the ***P'2*** 'direct' transfer will create a source term that will then be subject to ***P3***, after vertical homogenisation inside the layer.

$$P2 = P'2 + (1 - P'2)P3$$

But this is a biased answer, because the corresponding outgoing part will create a sink that will lower the part of the effect proportional to ***1-P'2***. So the multiplier will be, in first approximation, ***P3-P3.P3***. But then there is again a source term and hence we shall have ***P3-P3.P3+P3.P3.P3*** as multiplier, etc. One finally gets, with the compact expression for an infinite series:

$$P2 = P'2 + (1 - P'2)P3/(1 + P3) \quad \Rightarrow \quad P2 = \frac{P'2 + P3}{1 + P3}$$

This expression (the only physically 'true' one) does warrant ***P2>P3*** only for ***Z*** smaller than ***~0.96*** but, for bigger ***Z*** values, the difference between the two is always very small (maximum distance of ***0.015*** for ***Z*** about ***2.4***). However, for the stationarity thinking to work, we need the mean Courant number to be big enough for making the vertical redistribution hypothesis (central to the use of a budget equation for an assumed homogeneous q_r) plausible. Below a certain value (***~1.04*** here) this is not any more the case. This may of course be considered as an intrinsic weakness of the statistical sedimentation proposal, but any advective method would suffer from a similar drawback (the advected species may still be in the top of the layer and would have to be vertically redistributed at the end of the time-step). Here we simply can quantify this drawback and say for which values of ***Z*** it starts to be present and how small it remains in terms of differences between ***P2*** and ***P3***. Of course, it would have been more satisfying if the limiting CFL value had been below one (the layer-crossing-limit for drops having the average fall-speed), but the above exact number (***~1.04***) is sufficiently close to one for the argument to be valid for an anyhow non-homogeneous spectrum of fall-speeds.

Appendix 4: Kessler-type method for the link between rainfall rate and mean fall speed

We are still in the same framework as for Appendix 1. This time we restart from

$$a = a_0\,(\rho_0 / \rho)^{\alpha}$$

The computation of the mass-weighted average fall speed is simply:

$$\overline{w} = \frac{\int_0^{\infty} aD^{\alpha} N_0 e^{-\lambda D} \frac{\pi D^3}{6 v_l} dD}{\int_0^{\infty} N_0 e^{-\lambda D} \frac{\pi D^3}{6 v_l} dD} = a \frac{\Gamma(\alpha + 4)}{\Gamma(4) \lambda^{\alpha}}$$

with the rainfall rate given by

$$R = \frac{\Gamma(\alpha + 4) a N_0 \pi}{6 v_l\, \lambda^{\alpha + 4}}$$

Hence, eliminating once again $\boldsymbol{\lambda}$ we have:

$$\overline{w} = \frac{1}{\Gamma(4)} \left[a_0 \Gamma(\alpha + 4) \right]^{\frac{4}{4+\alpha}} \left[\frac{6 v_l R}{\pi N_0} \left(\frac{\rho_0}{\rho} \right)^4 \right]^{\frac{\alpha}{4+\alpha}}$$

This time it is $\boldsymbol{\alpha}$ which is approximated (to 0.8) and $\boldsymbol{a_0}$ is recomputed to deliver the same speed for a 3 mm diameter at half the surface pressure in the standard atmosphere. All this leads to

$$\overline{w} = \Omega^{r} \left(\frac{R}{\rho^4} \right)^{1/6} \qquad\qquad \Omega^{r} = 13.4 \ (SI\ units)$$

<u>Appendix 5</u>: Kessler-type method for the cloud water collection by rain

With the hypothesis of uniform collection by crossed volume, with the addition of an efficiency factor $\boldsymbol{E_{ff}}$, the basic equation for one drop writes:

$$\frac{dM}{dt} = E_{ff} \frac{\pi D^2}{4} a D^{\alpha} \rho \, q_l$$

this leading to

$$-\frac{dR}{dz} = \int_0^{\infty} N_0 e^{-\lambda D} E_{ff} \frac{\pi D^2}{4} a D^{\alpha} \rho \, q_l \, dD$$

equivalent to

$$\frac{dR}{dp} = \frac{E_{ff} \pi a N_0 \Gamma(\alpha+3)}{4 g \lambda^{\alpha+3}} q_l$$

with the rainfall rate still given by

$$R = \frac{\Gamma(\alpha+4) a N_0 \pi}{6 v_l \, \lambda^{\alpha+4}}$$

Hence, applying the same type of manipulation as in Appendix 1 finally gives

$$\frac{dq_l}{dt} = -\frac{\Gamma(\alpha+3) E_{ff} \pi a N_0}{4} \left(\frac{6 v_l}{\Gamma(\alpha+4) a N_0 \pi} R \right)^{\frac{\alpha+3}{\alpha+4}} q_l = -C E_{ff} R^{\frac{\alpha+3}{\alpha+4}} q_l$$

Here one rounds $\boldsymbol{\alpha}$ to 1 and one neglects in the final formula the density dependency through $\boldsymbol{a}$ (in power 1/5 only). There are two possible ways of rescaling, once we can use size or flux indifferently: either like in Appendix 4, which gives $\boldsymbol{C}$=0.318; or by replacing the size constraint by a flux one of 5mm/hour, which gives $\boldsymbol{C}$=0.357. $\boldsymbol{E_{ff}}$ is taken equal to 0.2 according to Lopez (2002). We thus finally use:

$$\frac{dq_l}{dt} = -C_E^r \, . R^{4/5} \, . q_l \qquad\qquad C_E^r = 6.7 \, 10^{-2}$$

Appendix 6: The WBF process treated as a third kind of auto-conversion

We follow here van der Hage (1995) and part of the analysis Gerard (2007) made out of it.

The basic idea is to parameterise the WBF process as a direct transfer from liquid cloud droplets to snowflakes, the intensity depending on the quantity of cloud ice-crystals as well, but without interference of the raindrops. This differs from Gerard's approach who considers a general enhancement of both 'classical' auto-conversion processes, but that method seems closer to the idea of van der Hage, even if the latter is not very explicit about the technical implementation of his theoretical proposal.

The basis of the mathematical treatment is to rely on the ratio of the water depletion rates both by 'classical' auto-conversion and by the WBF process. It is expressed in terms of number of droplets in van der Hage and, alike Gerard, we assume that this is directly transferable to specific amount quantities (no difference in the spectral selectivity of both processes).

This writes:

$$\left(\frac{dq_l}{dt}\right)_{WBF} \Big/ \left(\frac{dq_l}{dt}\right)_{ACO} = G.N_i / N_d$$

with a gain factor $\boldsymbol{G}$ and $\boldsymbol{N_{i/d}}$ the number of ice-crystals and droplets in the cloud, respectively.

Introducing the mean radius of ice-crystals and droplets as well as the expression of Equation (19) of van der Hage for G, we obtain the proportionality relationship:

$$\left(\frac{dq_l}{dt}\right)_{WBF} \Big/ \left(\frac{dq_l}{dt}\right)_{ACO} \propto G\left(q_i / q_l\right) / \left(r_i / r_d\right)^3 \propto \left(q_i / q_l\right)\left(\frac{r_i\left[c_i s_i\right]}{r_d^3}\right) / \left(r_i / r_d\right)^3$$

The bracketed expression can be taken proportional to the product of the proportions of ice and liquid water within the cloud (see Fig. 2 of van der Hage), the process being most active when both proportions are equal. Still according to van der Hage, the WBF process matters with respect to the classical auto-conversion mostly for small droplets. We shall convert this idea in the use of the asymptotic behaviour near zero of the Sundquist-type formulae for all types of auto-conversion. Since a tuning of the general multiplying constant $\boldsymbol{F^a_{WBF}}$ will anyhow be necessary, this does not seem to be too constraining a hypothesis (Gerard, who uses the discontinuous Kessler type of auto-conversion formula for the 'classical' part, encounters a different problem at that stage and thus parameterises the final result in a quite different way). The way of treating the problem proposed here is also consistent with the idea that it is $\boldsymbol{q_l}$ which will be depleted while it would disappear from the equation and be replaced by $\boldsymbol{q_i}$ if we would base our analysis on the other asymptotic behaviour. Our choice leads to:

$$\left(\frac{dq_l}{dt}\right)_{WBF} \propto \left[\frac{q_l^3}{\tau_l\left(q_l^{cr}\right)^2}\right](q_i/q_l)\frac{q_l.q_i}{\left(q_l+q_i\right)^2}/r_i^2$$

$$=\left[\frac{q_l}{\tau_l}\frac{q_l.q_i}{\left(q_l^{cr}.q_i^{cr}(T)\right)}\right]\frac{q_l.q_i}{\left(q_l+q_i\right)^2}\left[\frac{q_i^{cr}(T)}{r_i^2(T).q_l^{cr}}\right]$$

After some rescaling of its lower case parameters, the first bracket can be rewritten in the general Sundquist way. The (***1/τ_l***) rescaling can at that stage be merged with the general tuning of the intensity of the WBF process (Equation (19) of van der Hage gives only a lower limit to the G value). The only remaining problem is therefore the parameterisation of the second bracket. The best we can do is to consider it as a constant, also entering the global tuning of the whole effect. This amounts to assume a parallel decrease of the critical threshold for classical auto-conversion and of the mean surface of ice-crystals, a not too daring hypothesis.

We can now abandon the proportionality equations and symbolically write the expression already given in the main text:

$$\left(\frac{dq_l}{dt}\right)_{WBF} = -F_{WBF}^{a}\frac{q_l}{\tau_l}\frac{q_l.q_i}{(q_l+q_i)^2}\left(1-e^{-\frac{\pi}{4}((q_l.q_i)/(q_l^{cr}.F_{WBF}^{b}.q_i^{cr}(T).F_{WBF}^{b})}\right)$$

At that stage we made the hypothesis that both critical values are scaled in the same way (but given the chosen formula this is purely a choice of presentation) and we are left with the problem of evaluating the two constants $\boldsymbol{F^a_{WBF}}$ and $\boldsymbol{F^b_{WBF}}$. The task is not an easy one, but we do have a few hints:

- the figures for $\boldsymbol{G}$ and $\boldsymbol{r_i}$ given by van der Hage, some knowledge about usual values of the critical thresholds and Gerard's tunings gives us $\boldsymbol{F^a_{WBF}/(F^b_{WBF})^2}$ somewhere between ***16*** and ***40***;
- we know that $\boldsymbol{F^b_{WBF}}$ must be big enough not to allow returning to the linear asymptotic behaviour before the WBF process ceases to be dominant (with our new expression it is $\boldsymbol{q_i}$ which otherwise disappears then from the expression, apart from the 'product of proportions' factor);
- this is especially true since the latter factor forces first $\boldsymbol{q_i}$ to become of the order of $\boldsymbol{q_l}$ for the WBF effect to be meaningful, while the classical thresholds differ by roughly one order of magnitude;
- despite the fact that the total effect probably saturates for high values of $\boldsymbol{q_l}$, $\boldsymbol{F^a_{WBF}}$ should not be overwhelmingly big for the case the linear asymptotic behaviour is reached after all.

It is of course impossible to fulfil all constraints at the same time, but $\boldsymbol{F^a_{WBF}=300}$ and $\boldsymbol{F^b_{WBF}=4}$ seem to be a good compromise solution, the first value being of course far more uncertain than the second one.

<u>Appendix 7</u>: Temperature dependencies, especially for the various collection processes

One takes for granted the following:

- The formulae B3 and B4 of Lopez (2002) are here treated in the spirit of ***R/S*** replacing ***ρ.q*** $\boldsymbol{\rho . q_{r/s}}$ under the power ***1/6***;
- The collection efficiency concerning ice crystals is $\boldsymbol{f_{l/i}*(T)/5}$ in case of rain drops;
- The collection efficiency concerning cloud droplets is ***1/5*** in case of rain drops;
- The same 'dependencies' apply in the case of snow flakes but they are multiplied by three factors: (i) a ***0.5*** multiplication at ***0°C*** (snow flakes are intrinsically less good catchers than rain drops, see Lopez (2002)); (ii) a geometry factor (snow flakes scan a bigger surface at equal other properties) that can be estimated from Lopez (2002) figures at ***15.85***; (iii) a functional dependency on temperature that we shall now seek.

The dependency of the collection constant $\boldsymbol{C_E}$ upon $\boldsymbol{N_0}$ and ***a*** is of the type (see Appendix 5) $(a.N_0)^{1/5}$ while the dependency already encapsulated in the value of ***Ω*** is of the type (see Appendix 4) $(a^5/N_0)^{1/6}$.

Let us assume for the time being that the $\boldsymbol{c_t}$ value is yet unknown for the ***Ω*** dependency. On the contrary, we accept the ***-0.1222*** value of Lopez (2002) for the dependency of $\boldsymbol{N_0}$ for snow (the lower the temperature, the more there are flakes).

This gives us the following proportionality rules:

$$(a.N_0)^{1/5} \propto \left(\frac{a^{5/6}}{N_0^{1/6}}\right)^{6/25} .\left(N_0\right)^{6/25} \propto e^{(6/25).(c_t-0.1222)(T-T^*)}$$

which we wish (in order to keep a single absolute value for the exponents) to be proportional to

$$e^{-c_t(T-T_t)}$$

This is obtained for $\boldsymbol{c_t=0.1222\times(6/31)\approx 0.02365}$, a value indeed rather close to $\boldsymbol{c_t*}$.

What will then be the collection efficiency factor for snow at ***0°C***? One has to take into account the ***1./3.959*** factor on ***Ω***, the ***0.25*** factor on $\boldsymbol{N_0}$, the above choice for the exponents and the intrinsic ***0.5*** and ***15.85*** ratios on top of everything. This gives a bulk ratio of

$$\frac{C_E^s}{C_E^r} = 7.925\times(0.25/3.959)^{6/25} = 4.085 \;\Rightarrow\; C_E^s = 4.085\times 6.710^{-2} \approx 2.7410^{-1}$$

In summary we have the following equations for all four kinds of collections:

$$\left(\frac{dq_l}{dt}\right)_R = -C_E^r . R^{4/5} . q_l$$

$$\left(\frac{dq_i}{dt}\right)_R = -C_E^r . R^{4/5} . q_i \times f_{s/i}^*(T)$$

$$\left(\frac{dq_l}{dt}\right)_S = -C_E^s . S^{4/5} . q_l / f_{s/i}^*(T)$$

$$\left(\frac{dq_i}{dt}\right)_S = -C_E^s . S^{4/5} . q_i$$

In the fourth case, both temperature dependencies cancel each other (given our choice of having a single c_t*) and the snow/ice effect is simply about four times more intense than the rain/water one (the first case), independently of the air temperature value. Remember however that the fall speed effect on the ***R*** and ***S*** fluxes is about three, so that, when expressed in terms of $q_{r/s}.q_{l/i}$ (alike for Lopez) the snow/rain ratio of collection factors is only about ***1.36***, to be compared with the ***1.78*** value of Lopez (2002), both of course only valid at ***0°C***.

All other three temperature-dependencies (ice-snow auto-conversion efficiency ($1/\tau(T)$), ice-snow auto-conversion threshold ($q_i^{cr}(T)$), snow fall speed) are of the multiplying type for $f_{l/i}^*(T)$. In the latter case this gives:

$$\Omega^s = 3.4\, f_{l/i}^*(T)$$

Appendix 8: Introduction of a (non-prognostic) pseudo-graupel effect

One elects to influence here the results of the computations for all fluxes and pseudo-fluxes related to the ice-phase (up to now indifferently mentioned as snow) and to the water-phase of the falling precipitation (which will both still be the only influential output of APLMPHYS) in a single manner. The graupel effect is synthesised in the ratio r_g between a pseudo-graupel flux and the total 'snow' flux and r_g just influences the averaged properties of the fall speed and of the collection efficiency for the falling ice-phase:

$$\left[\Omega^r / \overline{\Omega}^s\right] = r_g + (1 - r_g).\left[\Omega^r / \Omega^s(T)\right]$$

$$\left[C_E{}^r / \overline{C}_E{}^s\right] = r_g + (1 - r_g).\left[C_E{}^r / C_E{}^s(T)\right]$$

The computation of the pseudo-graupel flux that will lead at the bottom of one given layer to the r_g ratio that will be used in the layer just below for the averaging above-described computations follows the following principles:

- No direct reliance on any prognostic quantity, in order to improve numerical stability.
- The statistical sedimentation functions for the pseudo-graupel are always those of the liquid phase. P2 multiplies the flux at the top of the layer, passed from the bottom of the layer above. P1 multiplies an estimate of q_g, obtained at the bottom of the above layer through the division of the pseudo-graupel flux by air density times the fall speed of the liquid phase, the latter being recomputed with all the available information at the top and bottom of the said layer. P3 multiplies the local evolution contributions which will now be detailed a bit more.
- Only the WBF part of the auto-conversion contributes to the pseudo-graupel flux.
- The collection for the ice-phase part is first computed with the above-described average efficiency. Then the proportion attributed to the pseudo-graupel flux is reduced with respect to r_g (in order to account for the lower relative collection efficiency) following:

$$r_g' = \frac{r_g}{r_g + (1 - r_g).C_E^s(T) / C_E^r}$$

- The ice-phase evaporation and ice-phase melting along the precipitation fluxes' path is first computed globally like in ACPLUIE, with the above-described average fall-speed. Then, similarly to the previous point, the part attributed to the pseudo-graupel flux is diminished with respect to r_g (in order to account for the higher relative fall speed) following:

$$r_g'' = \frac{r_g}{r_g + (1 - r_g).\sqrt{1 - m_e\,(1 - \Omega^r / \Omega^s(T))}} \quad \text{(see Part A)}$$

- The freezing of water along the precipitation fluxes' path is computed like in ACPLUIE and all attributed to the pseudo-graupel flux.

- At the bottom of the layer both the pseudo-graupel flux and $\boldsymbol{q_g}$ are forced to remain smaller or equal than their total ice-phase counterparts.

<u>Appendix 9</u>: Taking into account the geometry of clouds and falling precipitations

One should here distinguish between the basic principles (all nearly self-constrained within the environment of the APLMPHYS code) and their concrete application for the present version of the said code.

<u>The principles</u>:

- One profits from the statistical sedimentation algorithm to do all computations locally. The flux reorganising steps can then take place just when changing layer in the vertical. They are done either for random cloud overlap (LRNUMX=.F.) or for maximum overlap of adjacent clouds together with random overlap of clear air separated parts (LRNUMX=.T.).
- Each layer is divided in four parts. The three we are interested in are the top-seeded part of the cloud, the non-top-seeded part of the cloud and the precipitation covered part of the clear air fraction.
- The auto-conversion calculations are only called once; the collection calculations are called twice (top-seeded and non-top-seeded parts separately); the evaporation-sublimation plus melting-freezing calculations are called twice (precipitation covered clear air and averaged cloudy part –after homogenisation of the collection effect-, the latter only for the melting/freezing effect only of course, even if simplicity of the code requires to go through the same computing loops).
- When dealing with the pseudo-graupel effect, the collection efficiency used for the non-top-seeded part is of course simply $\boldsymbol{C^{s}{}_{E}(T)}$.
- The ***P0/P1/P2/P3*** functions for the statistical sedimentation are assumed to be the same everywhere. The water vapour saturation deficit is concentrated in the non-cloudy part of the mesh and is assumed homogeneous there. The $\boldsymbol{q_l}$ and $\boldsymbol{q_i}$ variables are assumed homogeneous within the cloud. The $\boldsymbol{q_r}$ and $\boldsymbol{q_s}$ variables are depending whether one is in the cloudy or clear air part of the mesh; they are computed proportionally to the local intensity of the precipitation flux, in accordance with the unique choice of the ***Pn*** functions, which implies a unique mean fall velocity.
- The cloudy part is supposed to make its output at the bottom of the layer homogeneous.
- The computations of the fractions and flux-intensities at the top of the next layer are done at the very end of the vertical loop. They are trivial in the case LRNUMX=.F. (linear recombining only) but rather complicated in the opposite case (see below).

<u>Rough sketch of the application in the (recommended) case LRNUMX=.T.</u>

- Like for radiation, we assume maximum overlap of adjacent cloudy layers and random positioning of cloudy parts separated by clear air. This is applied strictly for the cloudy parts. For the rain-covered areas, there is a bit of arbitrariness in applying the same rule, on top of the choice already made for clouds. The solution advocated here is one that allows both physical consistency and a single formula for each recombination (see the equations below). Other more complex (and may be more physical) solutions could be developed.

- Let us name ***N**** the cloud cover of the layer the rain is leaving and ***N*** the same for the one it is entering. Let us have (also with the '*' convention) $\boldsymbol{Pr_o}$ and $\boldsymbol{Pr_e}$ for the 'seeded' proportions in the cloudy and clear air parts respectively and $\boldsymbol{Fi_o}$ and $\boldsymbol{Fi_e}$ for the flux intensities (outgoing with '*' and incoming without it, thus).

- The equations which are covering all possible situations are:

$$\mathrm{Pr}_o = \left[\min(N, N^*) + (N - \min(N, N^*))\,\mathrm{Pr}_e{}^*\right]/N$$

$$\mathrm{Pr}_e = \left[(\max(N, N^*) - N) + (1 - \max(N, N^*))\,\mathrm{Pr}_e{}^*\right]/(1 - N)$$

$$Fi_e = Fi_e{}^* \qquad if \qquad N > N^*$$

$$Fi_o = Fi_o{}^* \qquad if \qquad N < N^*$$

  and, for the last two expressions, the other $\boldsymbol{Fi_{o/e}}$ is deduced from the knowledge of the total precipitation flux and from ***N*** and $\boldsymbol{Pr_{o/e}}$.